Rayleigh waves in a thermo-viscoelastic solid loaded with viscous fluid of varying temperature

J.N. Sharma, R. Sharma and P.K. Sharma
Department of Mathematics, NIT Hamirpur, (HP) INDIA

ABSTRACT: In the present paper is concerned with study of thermoelastic interaction in an infinite Kelvin-Voigt type viscoelastic, thermally conducting solid bordered with viscous liquid half-spaces/layers of varying temperature. Complex secular equations in closed and isolated mathematical conditions for Rayleigh wave propagation in completely separate terms are derived. The results for coupled and uncoupled theories of thermoelasticity have been obtained as particular cases from the derived secular equations. In order to illustrate and compare the theoretical results, the numerical solution is carried out for copper material by using the functional iteration method. The computer simulated results in respect of dispersion curves, attenuation coefficients, amplitudes of temperature change and displacements are presented graphically.

Keywords: Kelvin-Voigt model, Rayleigh waves, viscoelastic, attenuation

INTRODUCTION

The study of viscoelastic behavior in bone and in bioprotective materials is of interest in several contexts. Materials used for structural applications of practical interest may exhibit viscoelastic behavior which has a profound influence on the performance of that material. For example, viscoelastic shoe insoles are useful in reducing mechanical shocks transmitted to the bones and joints. Materials used in engineering applications may exhibit viscoelastic behavior as an unintentional side effect. Viscoelasticity is of interest in materials science, metallurgy, and solid state physics since it is causally linked to a variety of microphysical processes and can be used as an experimental probe of those processes. The casual links between viscoelasticity and microstructure are exploited in the use of viscoelastic tests as an inspection or diagnostic tool. Viscoelasticity was examined in the late twentieth century when synthetic polymers were engineered and used in a variety of applications.

The stress response of these materials depends on both the strain applied and the strain rate at which it was applied. Because of the time dependent material behavior of viscoelastic materials, their behavior is dependent on the history of loading and are said to have a 'memory'. The linear theory of thermo-viscoelasticity describes the linear behavior of both elastic and anelastic materials and provides a basis for describing the attenuation of seismic waves. Before 1960, most of the work on linear viscoelastic wave propagation for which explicit solutions have been obtained was essentially two dimensional. Bland [1] has given an account of three dimensional linear viscoelastic theory. He concluded that as in perfectly elastic isotropic medium under assumption of small displacement, two types of waves can be propagated in an isotropic viscoelastic medium when body forces are absent. The first type of wave called as dilatational or longitudinal or \( P \)-type wave, the second type of waves, known as shear or equivoluminal or \( S \)-type wave. Viscoelastic materials are of great importance in industry and therefore have attracted a lot of interest in the engineering community. On the other hand, mathematicians often find viscoelasticity interesting because of its present applications and challenges for the theory of integro-differential equations. Gurtin and Sternberg [2] discussed the linear theory of viscoelasticity.

The basic governing equations of thermoelasticity in the usual framework of linear coupled thermoelasticity consist of the wave type (hyperbolic) equations of motion and the diffusion type (parabolic) equation of heat conduction. Lord and Shulman [3], Green and Lindsay [4] and Green and Laws [5] have formulated generalized dynamical theories of thermoelasticity in which the equation of heat conduction is also hyperbolic. Some researchers such as Ackerman et al. [6], Ackerman and Overtone [7], Guyer and Krumhansl [8], proved experimentally for solid Helium that thermal waves (second sound) propagating with finite, though quite large speed, also exist although for most of the solids the corresponding frequency window namely, range of the frequency of thermal excitations in which thermal waves can be detected, is extremely limited. Green [9] proved the uniqueness of the equations derived by Green and Lindsay and studied the propagation of acceleration waves.

The existence of Rayleigh waves was predicted by Lord Rayleigh [10] in 1885 for whom they were named. They are distinct from other types of seismic waves, such as \( P \)-waves and \( S \)-waves or Love waves and travel the surface of a relatively thick solid material penetrating to a depth of one wavelength approximately. These waves combine both
Kurtze and Bolt [15] derived the bulk modulus of the liquid, \( L \), and \( c_{13}^L \). To study the surface waves \( \rho \kappa \) are the density and longitudinal velocity of \( iL \) and \( Lii \). Plopa et al. [16] studied Rayleigh and Lamb waves at liquid-solid boundaries. Wu and Zhu [17] studied the propagation of Lamb waves in a plate bordered with inviscid liquid layers on both sides.

Banerjee and Kundu [18] investigated ultrasonic field modeling in plates immersed in fluid. Bossart et al. [19] gave hybrid numerical and analytical solutions for acoustic boundary problems in thermo-viscous fluids. Craster and Williams [20] derived a reciprocity relation for fluid-loaded elastic plates that contain rigid defects. Wang and Zhang [21] applied boundary element method for simulating the coupled motion of a fluid and a three-dimensional body. Skelton [22] studied line force receptance of anelastic cylindrical shell with heavy exterior fluid loading. Krishna and Ganesan [23] studied the fluid filled and submerged cylindrical shells with constrained viscoelastic layer. Nayfeh and Nagy [24] derived the exact characteristic equations for leaky waves propagating along the interfaces of several systems involving isotropic elastic solids loaded with viscous fluids including semi spaces and finite thickness plates totally immersed in fluids or coated on one or on both sides by finite thickness fluid layers. The technique adopted by Nayfeh and Nagy [24] removed certain inconsistencies that unnecessarily reduced the accuracy and range of validity of Zhu and Wu [25] results. They suggested two possible models to improve the deficiency in the model used by Zhu and Wu [25]: (i) Modeling the viscous liquid as a hypothetical solid whose shear rigidity equals \( i\omega\mu_L \), and (ii) use of Stokes model which splits the viscosity parameter between \( c_{11} \) and \( c_{13} \). To study the surface waves in coated anisotropic medium loaded with viscous liquid, Wu and Wu [26] set \( c_{11} = \kappa + \frac{4}{3} \ i\omega\mu_L \), and \( c_{13} = \kappa - \frac{2}{3} \ i\omega\mu_L \), where \( \kappa = \rho_L c_L^2 \) is the bulk modulus of the liquid, \( \rho_L \) and \( c_L \) are the density and longitudinal velocity of the viscous liquid.

Hassan and Nagy [27] investigated the leaky Rayleigh waves in a fluid filled cylindrical cavity and found that the dispersive Rayleigh wave propagating around a concave cylindrical surface is substantially less attenuated by fluid loading than the corresponding wave on a flat surface. Garadzhaev [28] gave a note on the spectral theory of problems on normal oscillation of an ideal compressible fluid in rotating elastic shell. Wave propagation in a generalized thermoelastic solid cylinder of arbitrary cross-section immersed in a fluid is studied by Venkatesan and Ponnusamy [29]. Qi [30] studied the influence of viscous fluid loading on the propagation of leaky Rayleigh waves in the presence of heat conduction effects. Cho and Rose [31] investigated guided waves in a water loaded hollow cylinder. Popovics et al. [32] studied the propagation of leaky Rayleigh and Scholte waves at the fluid-solid interface subjected to transient point loading by using integral transform technique. Cherednichenko [33] studied the propagation of attenuated Rayleigh waves along a fluid-solid interface of arbitrary shape between a compressible fluid medium and an elastic solid. Mozhaev and Weihnacht [34] investigated subsonic leaky Rayleigh waves at liquid-solid interfaces.

theories of thermoelasticity. Sharma et al. [47] studied the propagation of Lamb waves in visco-thermoelastic plates under fluid loadings. Sharma and Pathania [48] investigated the propagation of generalized thermoelastic waves in anisotropic plates sandwiched between liquid layers.

**BASIC EQUATIONS AND CONSTITUTIVE RELATIONS**

The constitutive relations and basic governing equations of motion and heat conduction of generalized thermo-viscoelasticity for a Kelvin-Voigt type solid:

1. **Strain-Displacement relations**
   
   $$ e_{ij} = \frac{1}{2} (u_{i,i} + u_{j,j}) \quad (i, j = 1, 2, 3) \quad (1) $$

2. **Stress-Strain-Temperature Relations**
   
   $$ \sigma_{ij} = \lambda^* \delta_{ij} e_{kk} + 2 \mu^* e_{ij} - \beta^* (T + t_1 \delta_{kk} \hat{T}) \delta_{ij} \quad (i, j = 1, 2, 3) \quad (2) $$

3. **Equations of Motion**
   
   $$ \sigma_{ij} + b_i = \rho \ddot{u}_i \quad (i, j = 1, 2, 3) \quad (3) $$

4. **Equation of Heat Conduction**
   
   $$ K T_{ij,j} - \rho C_v (\dot{T} + t_0 \ddot{T}) = \beta^* T_{ii}(\dot{e} + t_0 \delta_{ii} \dot{e}) - (s + t_0 \delta) \quad (i, j = 1, 2, 3) \quad (4) $$

where

$$ \lambda^* = \lambda_e \left(1 + \alpha_0 \frac{\partial}{\partial t}\right), \mu^* = \mu_e \left(1 + \alpha_1 \frac{\partial}{\partial t}\right), $$

$$ \beta^* = (3 \lambda^* + 2 \mu^*) \alpha_T = \beta_c \left(1 + \beta_0 \frac{\partial}{\partial t}\right), $$

$$ \beta_e = (3 \lambda_e + 2 \mu_e) \alpha_T, \beta_0 = (3 \lambda_e \alpha_0 + 2 \mu_e \alpha_t) \alpha_T \beta_c \quad (5) $$

Here, $\ddot{u} (x, y, z, t) = (u, v, w)$ is displacement vector, $K$ is thermal conductivity, $C_v$ is specific heat at constant strain of the solid, $\rho$ is the density of medium, $\alpha_0$, $\alpha_1$, are the viscoelastic relaxation times, $T(x, y, z, t)$ is the temperature change, $e_{ij}$ and $\sigma_{ij}$ are the strain and stress tensors, $s$ is the dilatation, $\lambda_e$, $\mu_e$ are called Lame constants, $\alpha_T$ is the coefficient of linear thermal expansion, $\beta$ is thermomechanical coupling, $s$ is the heat source term, $b_i$ are the components of body force, $t_0$, $t_1$ are the thermal relaxation times, $T_0$ is the initial temperature of the medium, $\delta_{ij}$ is the Kronecker’s delta and $k = 1$ for LS theory and $k = 2$ for GL theory of generalized thermoelasticity. The comma notation is used for spatial derivatives and superposed dot is used for time differentiation.

The governing field equations in the context of linear generalized thermoelasticity can be written from equations 1-5 above. Also on setting thermal relaxation times $t_0=0=t_1$ and thermomechanical coupling $\beta=0$, the governing field equations for uncoupled thermoelasticity (elastic medium) can be obtained. The equations 1-5 are also subjected to appropriate boundary conditions consistent with the situations of various problems under investigation.

The basic governing field equations for viscous fluid (liquid) medium are given by

$$ \mu_L \frac{\partial}{\partial t} \nabla^2 \ddot{u}_L + \left( \lambda_L + \frac{1}{3} \mu_L \frac{\partial}{\partial t} \right) \nabla \nabla \ddot{u}_L = \beta_L \nabla \nabla^T \ddot{T}_L \quad (6) $$

$$ T_L = \frac{\beta^*_L T_0^*}{\rho_L C_V} \nabla \ddot{u}_L \quad (7) $$

where,

$$ \beta^*_L = 3 \lambda_L \alpha^*_T, \alpha^*_T \text{ being coefficient of volume thermal expansion; } \lambda_L \text{ is the Bulk modulus; } \rho_L \text{ is the density of fluid; } \mu_L \text{ is the kinematic viscosity of the liquid; } C'_V, T^*_0 \text{ are respectively the specific heat at constant volume and ambient temperature of fluid. Here } \ddot{u}_L \text{ is the velocity vector and } T_L \text{ is temperature deviation from the ambient temperature } T^*_0 \text{ of the fluid.} $$

**Boundary Conditions**

The boundary conditions at the solid-liquid interface to be satisfied are as follows:

(i) The magnitude of the normal component of the stress tensor of the plate should be equal to the pressure of the liquid.

(ii) The tangential components of stress tensor of plate and liquid should be equal.

(iii) The normal component of the displacement of the plate liquid should be equal.

(iv) The horizontal components of displacement of both the media should also be equal at the interface because of no slip condition.

(v) The thermal boundary condition is given by

$$ T_x + HT = 0 \quad (8) $$

where $H$ is the Biot’s constant. In the absence of liquid ($\rho_L = 0, \mu_L = 0$) that is for thermoelastic half space, the boundary condition corresponds to thermally insulated boundary for ($H \rightarrow 0$) and refers to isothermal one in case of ($H \rightarrow \infty$).
The present paper deals with the study of thermo-viscoelastic interactions in an infinite Kelvin-Voigt type viscoelastic solid bordered with viscous fluid with varying temperature. The Voigt—model of linear visco-thermo elasticity, earlier used by Kaliski [49] has been employed to consider the viscoelastic behavior of the solid. Relevant results of previous investigations, such as Graff [51], Achenbach [50] have been deduced as special cases. The generalized theory of thermo elasticity is employed to understand the effect of thermo mechanical coupling and mechanical relaxation time. Complex secular equations in closed and isolated mathematical conditions completely separate terms are derived. The results for coupled and uncoupled theories of thermoelasticity have been obtained as particular cases from the derived secular equations. In order to illustrate and compare the theoretical results in various situations, the numerical solution is carried out for copper material by employing the functional iteration method and the corresponding dispersion curves, amplitudes of displacements and temperature change are presented graphically.

FORMULATION OF THE PROBLEM

Consider a viscous liquid layer of finite thickness \( h \) overlying a homogeneous isotropic, thermally conducting viscoelastic solid in the undeformed state at uniform temperature \( T_0 \). We take origin of the coordinate system \( (x, y, z) \) at any point of the plane surface (interface) and the \( z \)-axis pointing vertically downward into the solid half-space which is thus represented by \( z \geq 0 \). We choose \( x \)-axis in the direction of wave propagation in such a way that all the particles on a line parallel to \( y \)-axis are equally displaced. Therefore, all the field parameters become independent of \( y \)-coordinate. Further, it is assumed that the disturbances are small and are confined to neighborhood of the interface \( z = 0 \) and hence vanish as \( z \to \infty \).

The basic governing equations of motion (3) and heat conduction (4) for the considered solid in the context of generalized thermo-viscoelasticity in the absence of body forces and heat sources, are

\[
(\lambda^* + \mu^*) \nabla (\nabla \cdot \vec{u}) + \mu^* \nabla^2 \vec{u} - \beta^* \nabla (T + t_1 \delta_{2k} \vec{T}_1) = \rho \ddot{\vec{u}}
\]

\[(10)

\[
K \nabla^2 T + \rho C_e \left( \dot{T} + t_0 \vec{T}_1 \right) = \dot{\beta} T_0 \nabla (\nabla \cdot \vec{u}) + t_0 \delta_{1k} \ddot{\vec{u}}
\]

\[(10)

where \( \vec{u} (x, z, t) = (u, 0, w) \) is the displacement vector and \( T(x, z, t) \) is the temperature change.

We define the quantities

\[
x' = \frac{\omega^*}{c_1} x' = \frac{\omega}{c_1}, T' = \frac{T}{T_0}, \xi' = \frac{\xi c_1}{\omega},
\]

\[
u' = \frac{\rho \omega^* c_1 u}{\beta T_0}, w' = \frac{\rho \omega^* c_1 w}{\beta T_0},
\]

\[
\varepsilon = \frac{\beta^2 T_0}{\rho C_e (\lambda^* + 2\mu^*)}, \omega' = \frac{\omega}{\omega}, \eta' = \frac{\omega' h}{c_1},
\]

\[
\sigma'_{ij} = \frac{\sigma_{ij}}{\beta^2 T_0}, \delta^2 = \frac{c_2^2}{c_1^2},
\]

\[
\alpha' = \omega \alpha \omega' = \omega' \alpha \}
\]

\[
c_2^2 = \frac{\mu c}{\rho}, c_1^2 = \frac{\lambda + 2\mu c}{\rho}, \omega' = C_e (\lambda^* + 2\mu^*)
\]

where \( \omega' \) is the characteristic frequency of the solid plate; \( \varepsilon \) is the thermomechanical coupling constant and \( c_1, c_2 \) are respectively, the longitudinal and shear wave velocities in the thermoelastic half-space.

In view of the quantities defined in equation (11) and introducing the displacement potential functions \( \phi \) and \( \psi \) of longitudinal and shear waves in the solid through the relations

\[
u = \phi_{,x} + \psi_{,x}, w = \phi_{,x} + \psi_{,x}
\]

in the governing equations 9-10, we obtain

\[
\left(1 + \alpha \frac{\partial}{\partial t}\right) \nabla^2 \psi = \frac{1}{\delta^2} \ddot{\psi}
\]

\[(13)

\[
\left(1 + \delta_{0} \frac{\partial}{\partial t}\right) \nabla^2 \phi - \left(1 + \beta_{0} \frac{\partial}{\partial t}\right) (T + t_0 \delta_{2k} \dot{T}) = \ddot{\phi}
\]

\[(14)

\[
\nabla^2 T - ( \dot{T} + t_0 \vec{T}_1 ) = c_T \left(1 + \beta_{0} \frac{\partial}{\partial t}\right) \nabla^2 \left( \ddot{\phi} + t_0 \delta_{1k} \ddot{\phi} \right)
\]

\[(15)

where \( \delta_0 = \alpha_0 + 2\delta^2 (\alpha_1 - \alpha_0) \).

In the liquid layers, the non-dimensional displacements are related to the scalar and vector velocity potentials through the relations

\[
u_L = \phi_{,x} + \psi_{,x}, w_L = \phi_{,x} - \psi_{,x}
\]

\[
\]

The governing equations in the fluid medium are given by

\[
\left[ \delta_{L}^2 \left(1 + \epsilon_L \right) + \left( \frac{4}{3} \psi_L \frac{\partial}{\partial t}\right) \right] \nabla^2 \psi_L - \ddot{\phi}_L = 0
\]

\[(17)

\[
\nabla^2 \psi_L - \psi_L = 0
\]

\[(18)

\[
T_L = \frac{-\epsilon_L \beta_L \delta_{L}^2}{\beta^2} \nabla^2 \ddot{\phi}_L
\]

\[(19)\]
where,
\[ c_L^2 = \frac{\lambda_L}{\rho_L}, \quad c_T^2 = \frac{\mu_L\omega^2}{\rho_L c_L^3}, \quad T_T = \frac{T_L}{T_0} \]
\[ \varepsilon_L = \frac{\beta_L^n T_0}{\rho_L c_L^3 \lambda_L}, \quad \beta = \frac{\beta_L}{\beta_e}, \quad \beta^* = 3\lambda_L \alpha^* \] (20)

**Boundary Conditions**

The boundary conditions at the solid-liquid interface \( Z = 0 \) to be satisfied are:

(i) The magnitude of the normal component of the stress tensor of the viscoelastic solid should be equal to the pressure of the liquid. This implies that
\[ \frac{\partial \phi}{\partial z} - 2 \left( 1 + \alpha_1 \frac{\partial}{\partial t} \right) (\phi_{xx} + \psi_{xz}) \]
\[ = -\frac{\rho_L}{\rho \delta} \left( \phi_L - 2v_L (\phi_{L,xx} + \phi_{L,zz} + \psi_{L,xx}) \right) \] (21)

(ii) The tangential components of stress tensor of viscoelastic solid and liquid should be equal, implying that
\[ \frac{\psi}{\delta^2} - 2 \left( 1 + \alpha_1 \frac{\partial}{\partial t} \right) (\psi_{xx} + \phi_{xz}) \]
\[ = -\frac{\rho_L}{\rho \delta^2} v_L (2\phi_{L,zx} + \psi_{L,zz} + \psi_{L,xx}) \] (22)

(iii) The normal component of the displacement of the solid should be equal to that of the liquid. This leads to
\[ \phi(z) - \phi(x, z) = \phi_L(z) - \psi_L(z, x) \] (23)

(iv) The horizontal components of displacement of both the media should also be equal at the interface because of no slip condition. This provides us
\[ \phi(x, z) + \psi(z, x) = \phi_L(x, z) - \psi_L(z, x) \] (24)

(v) The thermal boundary condition is given by Noda et al. (2000)
\[ T_L + H (T - T_L) = 0 \] (25)
where \( H \) is the Biot’s constant. In the absence of liquid (\( \rho_L = 0, \mu_L = 0 \)) that is for thermoelastic half-space, the boundary condition (25) corresponds to thermally insulated boundary for (\( H \rightarrow 0 \)) and refers to isothermal one in case of (\( H \rightarrow \infty \)).

**FORMAL SOLUTION**

We assume solution of the form
\[ f(x, z, t) = \bar{f}(z) \exp[i \xi(x - ct)] \] (26)
where \( f \) is any one of the functions \( \phi, \psi, T, \phi_L, \psi_L \) and \( c = \frac{\omega}{\xi} \) is the non-dimensional phase velocity, \( \omega \) is the frequency and \( \xi \) is the wave number. Upon using solution (26) in equations (13-15) and (17-20), we get a system of differential equations which give the expressions for \( \phi, \psi, T, \phi_L, \) and \( \psi_L \) upon satisfying the radiation condition (\( Re (m_k^2, \bar{\beta}) \geq 0, k = 1, 2 \)) as
\[ \phi = \sum_{k=1}^{2} A_k \exp(-\xi m_k^2 z + i\xi(x - ct)) \] (27)
\[ \psi = \sum_{k=1}^{2} A_k \exp(-\xi \bar{\beta}^* m_k^2 z + i\xi(x - ct)) \] (28)
\[ T = -\frac{i \omega^{-1} \tau_1 \tau_0^*}{\beta_e^*} \sum_{k=1}^{2} \xi^2 \left( \alpha_{k+1} - m_k^2 \right) \] (29)

For thermoelastic viscous solid half-space (\( 0 \leq z < \infty \)) and
\[ \phi_L = A_4 \sinh \xi \gamma_1 (z + h) \exp(i \xi(x - ct)) \] (30)
\[ \psi_L = A_4 \sinh \xi \gamma_2 (z + h) \exp(i \xi(x - ct)) \] (31)
\[ T_L = \xi^2 S_L A_4 \sinh \xi \gamma_1 (z + h) \exp(i \xi(x - ct)) \] (32)

For viscous liquid boundary layer (\( -h < z \leq 0 \)), here
\[ S_L = \frac{\frac{\varepsilon L \rho_L \delta_L^2}{\beta e} \left[ \frac{\delta_L^2 (1 + \xi L)}{4 \xi L v_L} \right] c^2 \alpha^2 + 1 - \frac{i \omega^{-1} c^2}{\delta_0} \] (33)
\[ \bar{\beta}^* = 1 - \frac{i \omega^{-1} c^2}{\alpha \delta^*}, \quad \gamma_2^* = 1 - \frac{i \omega^{-1} c^2}{\nu_L} \]
\[ \gamma_1^* = c^2 - \frac{\delta_0^2 (1 + \xi L)}{4 \xi L v_L}, \quad m_k^2 = 1 - \frac{i \omega^{-1}}{\delta_0} \omega_{k-1}^2, \quad k = 1, 2; \quad \tau' = \tau_0 \delta_1k + i \omega^{-1} \]
\[ \tau_0 = \tau_0 + i \omega^{-1}, \quad \tau_1 = \tau_0 \delta_2k + i \omega^{-1}, \quad \alpha_0 = \alpha_0 + i \omega^{-1}, \quad \bar{\beta}_0 = \beta_0 + i \omega^{-1} \]
\[ \delta_0^* = \delta_0 + i \omega^{-1}, \quad \alpha_1^* + a_2^* = 1 - i \omega \delta_0^\tau_0 + i \tau_1 \tau_0 e_{1 \omega}^3 \beta_L^2 \]
\[ a_1^2 a_2^* = \tau_0 \] (33)

**DERIVATION OF SECULAR EQUATION**

Invoking the interface boundary conditions (21-25) to at the surfaces \( z = 0 \) and using expression of \( \phi, \psi, T, \phi_L, \psi_L \) from (27-32), we get a system of five equations in five unknowns \((A_i, i = 1, 2, 3, 4, 5)\). The system of five linear equations so obtained will have non trivial solution if and
only if the determinant of the coefficients vanishes. After algebraic reductions and simplifications of the determinant, the secular relation governing the motion of non-leaky Rayleigh waves is obtained as

\[
\begin{align*}
\left\{ m_1^{*2} + m_2^{*2} + m_1^* m_2^* - \alpha^* \right\}(R_0 + a R_0^*) \\
- m_1^* m_2^* (m_1^* - m_2^*) (R_1 + a R_1^*) = 0 \\
H \left\{ (m_1^* + m_2^*) (R_0 + a R_0^*) - (\alpha^* + m_1^* m_2^*) \right\} \\
= \frac{H}{\xi} \left( R_1 + a R_1^* \right) = \frac{\delta_0^*}{\delta_0^*}
\end{align*}
\]

...(34)

where

\[
\begin{align*}
R_0 &= \rho (\omega - p^*) p^* T_4 T_5 \\
R_0^* &= -4 \rho^* + 4a + (p^* + 2i\omega \alpha_1^*) (\gamma_2^* - 1)\ B^* T_5 + T_4 T_5 R_0^* \\
R_1^* &= 2(\gamma_1^* - 2) p^* + \left\{ (p^* - \omega L) + 2a (\gamma_1^* - 2) \right\} (\gamma_2^* + 1) \\
R_1^* &= -4 \rho^* \omega^* \alpha_1^* - (\omega + 2i\omega \alpha_1^*) 2i\omega \alpha_1^* B^* T_4 T_5 \\
&\quad + \rho^* \gamma_4 T_5 R_1^* \\
R_1^* &= 4i\omega \alpha_1^* (\gamma_2^* - 2) \\
&\quad - \left\{ (\omega L + 2i\omega \alpha_1^*) - 2a (\gamma_2^* - 1) \right\} (\gamma_2^* + 1) \\
R_2^* &= T_4 (p^* + 2i\omega \alpha_1^*) (p^* + 2i\omega \alpha_1^* B^* T_5) \\
R_2^* &= T_4 (p^* + 2i\omega \alpha_1^*) (\gamma_2^* + 1) B^* T_5 - 2) \\
p^* &= -2i\omega \alpha_1^* - \frac{c^2}{\delta^2} T_4 = \frac{\tan \xi_1 h}{\gamma_1}, T_5 = \frac{\tan \xi_2 h}{\gamma_2}, \\
\omega_L &= \frac{c^2 \rho_L}{\rho \delta^2}
\end{align*}
\]

...(35)

This secular equation (34) contains complete information regarding phase velocity, attenuation coefficient and other characteristics of non-leaky Rayleigh waves. It is also noticed here that the secular dispersion relation for leaky Rayleigh waves in a thermo-viscoelastic solid bordered with viscous liquid half-space \((h \rightarrow \infty)\) can be obtained from the relation (34) by replacing \(\frac{\tan \xi_1 h}{\gamma_1} \rightarrow \frac{\xi_1}{\gamma_1}, \) \((k = 1, 2)\) respectively.

**PARTICULAR CASES OF DISPERSION RELATION**

This section is devoted to the reduction of the secular equation (34) under different situations such as

(i) **Coupled Thermoelasticity.** In the absence of the viscoelastic effect of solid, the mechanical relaxation times \(\alpha_0^* = 0 = \alpha_1^*\) and hence \(\alpha_0^* = \alpha_1^* = i\omega^{-1}, \) \(\delta_0^* = i\omega^{-1} = \beta_0^*\) and consequently, we have

\[
\begin{align*}
\alpha^* &= (1 - c^2) = \alpha^2, \quad \beta^* = \left( 1 - \frac{c^2}{\delta^2} \right) = \beta^2, \\
m_k^{*2} &= (1 - a_k^2) = \alpha_k^* = k = (1, 2).
\end{align*}
\]

Then the above secular equation (34) reduces to that of equation Sharma et al. (2008AMM). Further deductions are followed in the similar way and discussed in detail there.

(ii) **Viscoelasticity.** In case of uncoupled thermoelasticity, the thermal and mechanical fields get uncoupled and remain independent of each other \(\varepsilon_T = 0 = \varepsilon_L.\) Moreover, no heat transference will take place leading to \(H = 0.\) Consequently, the secular equation (34) reduces to

\[
\begin{align*}
\left( R_0^* + a R_0^* \right) - m_k^* \left( R_k^* + a R_k^* \right) = 0
\end{align*}
\]

...(36)

where the expressions of \(R_0^*, R_1^*, R_2^*, R_3^*, R_4^*, \) \(R_5^*, R_6^*\) are given by equation (35).

(iii) **Elasticity.** On setting \(\alpha_0^* = 0 = \alpha_1^*\) in equation, we obtain the transcendental Rayleigh frequency equations of wave propagation, which further reduces to that of Graff (1991) in the absence of fluid.

**SOLUTION OF THE SECULAR EQUATION**

The characteristic roots \(m_k^*, \) \((k = 1, 2)\) and \(\gamma_2\) defined by equations are complex and therefore, the wave number and phase velocity of the waves are complex quantities. Therefore, the waves get attenuated in space.

If we write

\[
c^{-1} = V^{-1} + i\omega^{-1} Q
\]

...(37)

such that \(\xi = R + iQ, R = \frac{\omega}{V}, \) where \(V\) and \(Q\) are real. The exponent in the plane wave solution becomes \(iR \left( x - V t \right) - Qx\) meaning thereby \(V\) is the propagation speed and \(Q\) the attenuation coefficient of the waves. Substituting equation (37) in secular equation (34) and other relevant relations, the phase velocity \((V),\) attenuation coefficient \((Q)\) and specific loss factor of energy dissipation can be computed at different values of \(R\) for the propagation of non-leaky Rayleigh waves by functional iteration method outlined below.
The secular equation (34) is of the form $F(c) = 0$ which upon using representation (37) leads to a system of two real equations $f(V, Q) = 0$ and $g(V, Q) = 0$. In order to apply functional iteration method we write $V = f'(V, Q)$ and $Q = g'(V, Q)$, where the functions $f'$ and $g'$ are selected in such a way that they satisfy the conditions

$$\left| \frac{\delta f'}{\delta V} \right| + \left| \frac{\delta f'}{\delta Q} \right| < 1, \quad \left| \frac{\delta g'}{\delta V} \right| + \left| \frac{\delta g'}{\delta Q} \right| < 1 \quad \text{...(38)}$$

For all $V, Q$ in the neighborhood of the root. If $(V, Q)$ be the initial approximation to the root, then we construct the successive approximations according to the formulæ

$$V_1 = f'(V_0, Q_0), \quad Q_1 = g'(V_1, Q_0)$$
$$V_2 = f'(V_1, Q_1), \quad Q_2 = g'(V_2, Q_1)$$

$$\vdots \vdots \vdots$$

$$V_{n+1} = f'(V_n, Q_n), \quad Q_{n+1} = g'(V_{n+1}, Q_n) \quad \text{...(39)}$$

The sequence $\{V_n, Q_n\}$ of approximations to the root will converge to the actual value $(V, Q)$ of the root provided $(V_0, Q_0)$ lies in the neighbourhood of the actual root. For the initial value $c = c_0 = (V_0, Q_0)$, the roots $m_i^*$ are computed from equation for each value of wave number $R$, for assigned frequency. The values of $m_i^*$ ($i = 1, 2$) so obtained are then used in secular equation (34) to obtain the current values of $V$ and $Q$ each time which are further used to generate the sequence (39). The process is terminated as and when the condition $|V_{n+1} - V_n| < \varepsilon, \varepsilon$ being arbitrarily small number to be selected at random to achieve the accuracy level, is satisfied. The procedure is continuously repeated for different values of the wave number $R$ to obtain corresponding values of the phase velocity $(V)$ and attenuation coefficient $(Q)$. The specific loss of energy $(SL)$ is given by (40) has also been computed.

**Specific Loss**

The ratio of energy dissipated $(\Delta W)$ in a specimen through a stress cycle, to the elastic energy $(W)$ stored in the specimen when the strain is maximum, is called ‘specific loss’. According to Kolsky (1963) in case of sinusoidal plane wave of small amplitude, the specific loss $\left(\frac{\Delta W}{W}\right)$ equals to $4\pi$ times the absolute value of imaginary part of $\xi$ to the real part of $\xi$ i.e. Hence here it is given by

$$\left(\frac{\Delta W}{W}\right) = 4\pi \left| \frac{VQ}{\omega} \right| \quad \text{...(40)}$$

**NUMERICAL RESULTS AND DISCUSSION**

In order to illustrate the theoretical results obtained in the previous section, some numerical results are presented. The material chosen for this purpose is copper, the physical data for which is given in Table 1 below. The values of non-dimensional thermal relaxation parameters ($\rho_0 = t_0 \omega^*, \mu_0 = t_0 \omega^*$) are taken as $t_1 = 0.3, t_0 = 0.5$. The value of Biot’s constant is taken as $H = 0.1$ for numerical calculations. The liquids chosen for the purpose of numerical calculations are light and heavy water, for which the velocity of sound and density are given by $c_s = 1.5 \times 10^3$ m/s, $\rho_s = 1000$ Kg m$^{-3}$ respectively. Upon using representation (37), the complex secular equation (34) is solved by using functional iteration method to obtain phase speed $(V)$ and attenuation coefficient $(Q)$ after computing complex characteristic roots.

<table>
<thead>
<tr>
<th>S. No.</th>
<th>Quantity</th>
<th>Units</th>
<th>Numerical Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$\rho$</td>
<td>Kg m$^{-3}$</td>
<td>$8.950 \times 10^3$</td>
</tr>
<tr>
<td>2.</td>
<td>$K$</td>
<td>Cal m$^{-1}$s$^{-1}$K$^{-1}$</td>
<td>$1.13 \times 10^2$</td>
</tr>
<tr>
<td>3.</td>
<td>$\varepsilon_T$</td>
<td>—</td>
<td>$0.00265$</td>
</tr>
<tr>
<td>4.</td>
<td>$\lambda_s$</td>
<td>Nm$^{-2}$</td>
<td>$8.2 \times 10^{10}$</td>
</tr>
<tr>
<td>5.</td>
<td>$\mu_s$</td>
<td>Nm$^{-2}$</td>
<td>$4.2 \times 10^{10}$</td>
</tr>
<tr>
<td>6.</td>
<td>$\omega^*$</td>
<td>sec$^{-1}$</td>
<td>$4.347 \times 10^{13}$</td>
</tr>
<tr>
<td>7.</td>
<td>$\alpha_T$</td>
<td>K$^{-1}$</td>
<td>$300$</td>
</tr>
<tr>
<td>8.</td>
<td>$\alpha_0 = \alpha_1$</td>
<td>sec</td>
<td>$1.0 \times 10^{-8}$</td>
</tr>
<tr>
<td>9.</td>
<td>$\alpha_0 = \alpha_1$</td>
<td>sec</td>
<td>$6.883 \times 10^{-8}$</td>
</tr>
</tbody>
</table>

The specific loss factor of energy dissipation has also been computed from equation (38). The computed simulated results in respect of various wave characteristics in the non-dimensional form corresponding to three situations of liquid loading namely, inviscid (ideal) liquid $(\rho_s \neq 0.0, \mu_s = 0.0)$, low viscous liquid $(\rho_s \neq 0.0, \mu_s = 0.1)$ and high viscous liquid $(\rho_s \neq 0.0, \mu_s = 1.0)$ have been plotted graphically in Figs.1 to 9 in case of GL-theory.

Fig.1 represents the variations of phase velocity profiles $(V)$ with respect to wave number $(R)$ on linear-log scales for ideal (inviscid), low viscous and high viscous types of fluid loading. Phase velocity profiles show decreasing trend in the wave number range $0 \leq R \leq 1.2$ in all the three considered cases of liquid loading with slight difference in their magnitudes before these become steady and stable afterwards. The variations of attenuation coefficient $(Q)$ versus with wave number $(R)$ have been plotted in Fig.2 in case of inviscid/viscous fluid loadings and free surface on linear-log scales. It is noticed that the profiles of this quantity in all the considered conditions prevailing at the interfacial surface increase monotonically in the wave number range $0 \leq R \leq 1$ and become steady and stable in case of free surface condition in contrast to fluid loaded one in which the profiles follow stable trends after observing fluctuating behavior for $R \geq 1$. It is also observed that the effect of inviscid fluid on this quantity is approximated between free surface and viscous fluid loading conditions.
loading is quite high as compared to rest of the cases of loadings, though it also becomes steady and stable alike other profiles at short wavelengths. The variations of phase velocity ($V$) with ambient liquid temperature ($T'_0$) are plotted in Fig.4. It is noticed that the phase velocity profiles in all the three considered cases of liquid loading have similar trend and behavior with liquid loading temperature except small differences in their magnitude. The magnitude in case of high viscous liquid loading is noticed to be nearly the double to that of inviscid/low viscous fluid. All profiles are dispersionless and follow nearly linear variations with temperature change of liquid.

The variations of attenuation coefficient ($Q$) versus ambient liquid temperature ($T'_0$) have been presented in Fig.5. It is revealed that profiles of this quantity pertaining to inviscid, low viscous and high viscous fluid loadings exhibit linear behaviour with increasing ambient temperature of the fluid and hence more or less dispersionless throughout the considered range of temperature. However, the magnitude of this quantity due to increasing viscosity is approximately one-third to that of inviscid fluid loading. Similar behaviour and trends of variations of various profiles of specific loss factor ($SL$) of energy dissipation have also been observed from Fig.6 except the magnitude of ($SL$) factor is approximately ten-thousand time to that of attenuation coefficient ($Q$) in Fig.5.

Fig.3 shows the variations of specific loss ($SL$) versus wave number ($R$) under free and inviscid/viscous fluid loading conditions. The profiles of this quantity corresponding to free and high viscous fluid loading conditions are more or less dispersionless at all wavelengths except in the wave number range $0 \leq R \leq 1$. However, the profiles of specific loss ($SL$) pertaining to inviscid and low viscous liquid loading at the interface follow dispersive behaviour. The degree of dispersion in case of inviscid fluid
It is also noticed while the magnitude of phase velocity is quite large in case of high viscous liquid loading as compared to that of inviscid/low viscous one at all values of the ambient liquid temperature \(T_0\) whereas the trend get reversed in case of attenuation coefficient \(Q\) and specific loss factor \(SL\). The variations of phase velocity \(V\), attenuation coefficient \(Q\) and specific loss factor \(SL\) versus liquid layer thickness \(h\) are plotted in Figs. 7 to 9. The profiles of all these quantities follow more or less linear variations and thereby exhibit dispersionless behaviour with increasing liquid layer thickness with small deviations in respect of attenuation and specific loss profiles at large values of the liquid thickness \(h\). Magnitude of attenuation and specific loss factor is quite small in case of high viscous fluid as compared to that of inviscid/low viscous fluid profiles in contrast to that of phase velocity in which this trend gets reversed.

**CONCLUSIONS**

Significant changes in the profiles of phase velocity, attenuation coefficient and specific loss factor of energy dissipation due to mechanical relaxation times have been observed in addition to the effects of different cases of considered fluid loadings. The behaviour of profiles pertaining to inviscid fluid loading has been observed to be approximated between those of free surface and viscous fluid loading profiles. The viscosity of both media results in decreasing the magnitude of attenuation and specific loss factor due to increasing liquid layer ambient temperature as well as thickness whereas it contributes in increasing the phase velocity of the interfacial waves. The liquid overlying the thermoviscoelastic solid halfspace has been successfully modelled as normal load (hydrostatic pressure) and thermal source simultaneously.

**REFERENCES**


[37] Shuvalov, A.L., Poncelet, O. and Deschamps, M., Analysis of the dispersion spectrum of fluid-loaded anisotropic plates: flexural-type branches and real-


