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A New Recursive Method for Solving State Equations Using Taylor Series

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ABSTRACT: The paper presents application of Taylor series to solve states of the control system. The systems are analyzed by first order Taylor series and as well as second order Taylor series. A new recursive method for the solution is discussed here. The state equation of a linear time invariant and differential equation of a nonlinear time invariant system are considered as example to clarify the method. The states of two systems are then solved by first order and second order Taylor series both and also compared with the exact solution which are reasonably close to each other. This comparison is shown in tables and graphs both to validate the new approach.

***Index Terms*—Linear time invariant, nonlinear time invariant, states, Taylor series.**

I. INTRODUCTION

The concept of Taylor series expansion was formally introduced by the English mathematician Brook Taylor in 1715. The concept, though quite old, has not lost its importance and is used in many areas of mathematical analysis.

Taylor's series is an expansion of a function into an infinite series of a variable t, or into a finite series plus a remainder term. The coefficients of the expansion or of the subsequent terms of the series involve the successive derivatives of the function.

In this paper Taylor series is utilized to solve dynamic differential equations [1] of a control system, e. g., state equations, to determine the states of the system. Several works have been done to form orthogonal operational matrix using Taylor series [3]. Time delay system has also been analyzed along with identification of parameters [4]. The presented method herein uses Taylor approximation to solve state as well as differential equations in a recursive manner.

Haar function [6], Walsh function [7], Block pulse function [8] are the mathematical tools used for state space problem solution. Later, hybrid functions [9, 10], triangular function [11] have been used extensively. But the Taylor series technique is much simpler and a powerful tool as well.

II. TAYLOR APPROXIMATION

A. First order Taylor approximation [2]

Consider a time function f(t) in an interval of width h, $t \in$ (ih, (i+1)h). A first order Taylor approximation $\bar{f}1(t)$ of the function f(t) around a point μ_i is represented as

$$\bar{f}1(t) \triangleq f(\mu_i) + \dot{f}(\mu_i)(t - \mu_i) \quad \ldots(1)$$

where $\mu_i \in \left(ih,\ (i+1)h\right)$

If the point μ_i coincides with the leading terminal point ih, then μ_i =ih and equation (1) becomes

$$\bar{f}1(t) = f(ih) + \dot{f}(ih)(t - ih) \quad \ldots(2)$$

If t=(i+1)h in (2), then

$$\bar{f}1\{(i+1)h\} = f(ih) + h\dot{f}(ih) \quad \ldots(3)$$

$\bar{f}1\{(i+1)h\}$ in equation (3) is the initial value of f(t) for the next interval $t \in \left((i+1)h,\ (i+2)h\right)$.

B. Second order Taylor approximation

A second order Taylor approximation $\bar{f}2(t)$ of the function f(t) around the same point μ_i is represented as

$$\bar{f}2(t) \triangleq f(\mu_i) + \dot{f}(\mu_i)(t - \mu_i) + \frac{1}{2!}\ddot{f}(\mu_i)(t - \mu_i)^2 \qquad \ldots(4)$$

With the assumption μ_i = ih, (4) reduces to

$$\bar{f}2(t) = f(ih) + \dot{f}(ih)(t - ih) + \frac{1}{2!}\ddot{f}(ih)(t - ih)^2 \qquad \ldots(5)$$

In (5), as before, we set t=(i+1)h then

$$\bar{f}2\{(i+1)h\} = f(ih) + h\,\dot{f}(ih) + \frac{h^2}{2!}\ddot{f}(ih) \qquad \ldots(6)$$

$\bar{f}2\{(i+1)h\}$ is the initial value of f(t) for the next interval $t \in ((i+1)h, (i+2)h)$.

III. SOLUTION OF STATE EQUATION VIA TAYLOR APPROXIMATION

A. Linear time invariant (LTI) system

Consider the state equation of a linear time invariant (LTI) system as

$$\dot{\mathbf{x}}(\mathbf{t}) = \mathbf{A}\mathbf{x}(\mathbf{t}) + \mathbf{B}\mathbf{u}(\mathbf{t}) \qquad \ldots(7)$$

and $\mathbf{x}(\mathbf{0}) = \mathbf{x_0}$

Differentiating (7), we have

$$\ddot{\mathbf{x}}(\mathbf{t}) = \mathbf{A}\dot{\mathbf{x}}(\mathbf{t}) + \mathbf{B}\dot{\mathbf{u}}(\mathbf{t}) \qquad \ldots(8)$$

For t=ih, equations (7) and (8) can be rewritten as

$$\dot{\mathbf{x}}(\mathbf{ih}) = \mathbf{A}\mathbf{x}(\mathbf{ih}) + \mathbf{B}\mathbf{u}(\mathbf{ih}) \qquad \ldots(9a)$$

$$\ddot{\mathbf{x}}(\mathbf{ih}) = \mathbf{A}\dot{\mathbf{x}}(\mathbf{ih}) + \mathbf{B}\dot{\mathbf{u}}(\mathbf{ih}) \qquad \ldots(9b)$$

Using (3) in equation (9), we can write following recursive equations

$\bar{\mathbf{x}}\mathbf{1}(\mathbf{h}) = \mathbf{x}(\mathbf{0})$ [redacted] $\mathbf{h}\,\dot{\mathbf{x}}(\mathbf{0})$

$\bar{\mathbf{x}}\mathbf{1}\{(\mathbf{i}+\mathbf{1})\mathbf{h}\} = \bar{\mathbf{x}}\mathbf{1}(\mathbf{ih})$ [redacted] $\mathbf{h}\,\dot{\bar{\mathbf{x}}}\mathbf{1}(\mathbf{ih})$ … (10)

where, i=1, 2, 3, …, N, N being a large number.

Thus, from (10), using (9a) and knowing **U(t)**, we can solve for the state vector **x(t)** recursively via first order Taylor approximation. Similarly, to obtain a more accurate recursive solution, we use second order Taylor approximation as given in (6) to get

$\bar{\mathbf{x}}\mathbf{2}(\mathbf{h})\} = \mathbf{x}(\mathbf{0})$ [redacted] $\mathbf{h}\,\dot{\mathbf{x}}(\mathbf{0}) + \frac{h^2}{2!}\ddot{\mathbf{x}}(\mathbf{0})$

$\bar{\mathbf{x}}\mathbf{2}\{(\mathbf{i}+\mathbf{1})\mathbf{h}\} = \bar{\mathbf{x}}\mathbf{2}(\mathbf{ih})$ [redacted] $\mathbf{h}\,\dot{\bar{\mathbf{x}}}\mathbf{2}(\mathbf{ih}) + \frac{h^2}{2!}\ddot{\bar{\mathbf{x}}}\mathbf{2}(\mathbf{ih})$ … (11)

where, i=1, 2, 3, …, N, N being a large number.

Thus, from (11), using both (9a) and (9b), we can solve for the state vector **x(t)** recursively via second order Taylor approximation.

Equations (10) and (11) may be solved with a fixed step size h or any dynamic step size, which means h can be changed during recursion.

B. Example

Consider the linear time invariant system

$$\dot{\mathbf{x}}(\mathbf{t}) = \begin{bmatrix} -1 & -1.8 \\ 5 & -1 \end{bmatrix}\mathbf{x}(\mathbf{t}) + \begin{bmatrix} 1.8 \\ 0 \end{bmatrix}\mathbf{u}(\mathbf{t}) \qquad \ldots(12)$$

and $\mathbf{x}(\mathbf{0}) = \mathbf{x_0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ with unit step input.

The exact solution is

$$x1(t) = 0.18 - 0.18\exp(-t)\cos 3t + 0.54\exp(-t)\sin 3t \qquad \ldots(13a)$$

and

$$x2(t) = 0.9 - 0.9\exp(-t)\cos 3t - 0.3\exp(-t)\sin 3t \qquad \ldots(13b)$$

Using equations (9a) and (10) we can solve for the states via first order Taylor approximation, and using equations (9a), (9b) and (11), we can solve the states with second order Taylor approximation. The results are given in Table 1 and Table 2. For computation, we have considered a time interval T = 2 s and the number of steps m = 20, so that h = 0.1 s.

Figure 1 shows the recursive solutions obtained via first order Taylor approximation while figure 2 shows the recursive solutions obtained using second order approximation.

Solution by 1st order Taylor series is represented by Taylor1 and 2nd order Taylor series by Taylor2 in graph.

To compare the credibility of the results, both the figures show the exact solution of the states x1 and x2. As expected, the second order approximation is way better than the first order approximation. In figure 3, all the results, i. e., recursive solution using first order Taylor approximation as well as second order Taylor approximation, and the exact solution, are shown together for better clarity.

If we gradually increase m, keeping the time interval T fixed (i. e., h is decreased), the first order Taylor approximation improves gradually. It is observed that for m = 160 the recursive solution almost overlaps the exact solution. This is shown in figure 4.

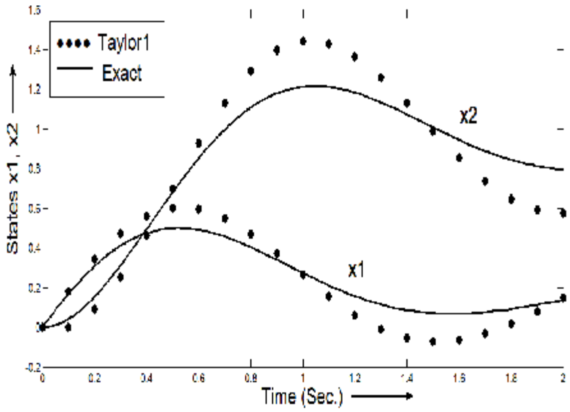

Fig. 1. Solution of the states x1 and x2 via first order Taylor approximation compared with the exact solution for m=20.

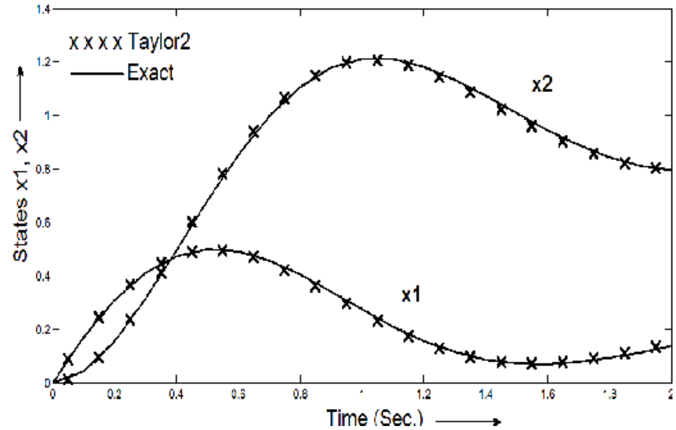

Fig. 2. Solution of the states x1 and x2 via second order Taylor approximation compared with the exact solution for m=20.

Table 1: Recursive solution of the state (x1) obtained via first order and second order Taylor approximation compared with the exact solution (for T=2 s, m=20 and h=0.1 s).

Time (Sec.)	X1		
	Exact solution x1(t)	Pointwise recursive solution of x1(t) with 1st order approximation	Pointwise recursive solution of x1(t) with 2nd order approximation
0	0	0	0
0.1000	0.1688	0.1800	0.1710
0.2000	0.3080	0.3420	0.3108
0.3000	0.4105	0.4716	0.4122
0.4000	0.4737	0.5591	0.4732
0.5000	0.4990	0.5999	0.4956
0.6000	0.4911	0.5947	0.4846
0.7000	0.4566	0.5485	0.4475
0.8000	0.4035	0.4700	0.3926
0.9000	0.3400	0.3705	0.3284
1.0000	0.2736	0.2618	0.2624
1.1000	0.2108	0.1558	0.2012
1.2000	0.1566	0.0629	0.1495
1.3000	0.1144	-0.0091	0.1104
1.4000	0.0857	-0.0550	0.0850
1.5000	0.0707	-0.0727	0.0731
1.6000	0.0682	-0.0634	0.0734
1.7000	0.0762	-0.0307	0.0835
1.8000	0.0921	0.0198	0.1006
1.9000	0.1131	0.0812	0.1218
2.0000	0.1362	0.1464	0.1445

Table 2: Recursive solution of the state (x2) obtained via first order and second order Taylor approximation compared with the exact solution (for T=2 s, m=20 and h=0.1 s).

Time (Sec.)	X2		
	Exact solution x2(t)	Pointwise recursive solution of x2(t) with 1st order approximation	Pointwise recursive solution of x2(t) with 2nd order approximation
0	0	0	0
0.1000	0.0418	0	0.0450
	0.1532	0.0900	0.1607
0.2000	0.3115	0.2520	0.3230
0.3000	0.4940	0.4626	0.5083
	0.6799	0.6959	0.6951
0.4000	0.8519	0.9262	0.8658
0.5000	0.9970	1.1309	1.0077
	1.1071	1.2921	1.1130
0.8000	1.1787	1.3979	1.1788
0.9000	1.2122	1.4433	1.2066
	1.2116	1.4299	1.2007
1.0000	1.1831	1.3648	1.1682
1.1000	1.1343	1.2598	1.1169
	1.0733	1.1292	1.0552
1.2000	1.0078	0.9888	0.9907
1.3000	0.9444	0.8536	0.9299
1.4000	0.8886	0.7365	0.8778
	0.8439	0.6475	0.8374
1.5000	0.8123	0.5926	0.8105
1.6000	0.7944	0.5740	0.7968

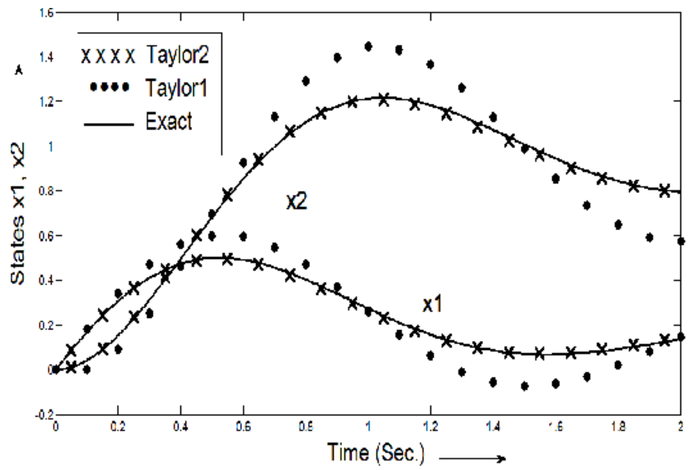

Fig. 3. Comparison between first order approximation and second order approximation for m = 20.

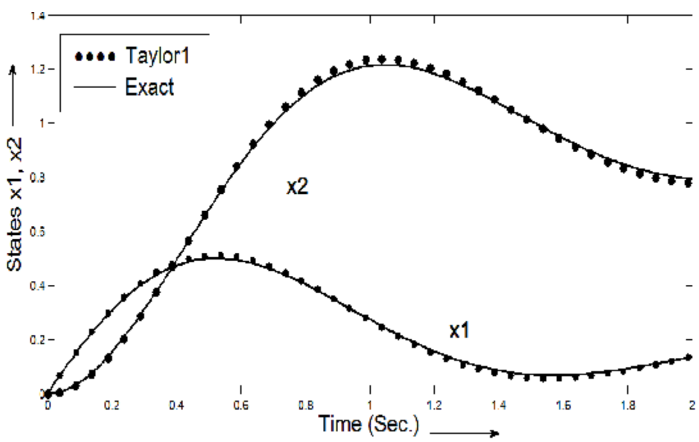

Fig. 4. Taylor approximation upto first order approximation and for m = 160.

A. Non-linear time invariant (NLTI) system

Consider the following nonlinear differential equation

$$f\dot{x}+kx+k'x^3=0 \quad \ldots (14)$$

where, f, k and k' are constants.

Let the initial state be $x(0)=x_0$

Differentiating (14), we have

$$f\ddot{x}=-k\dot{x}-3k'x^2\dot{x} \quad \ldots (15)$$

For t=ih, equations (14) and (15) can be rewritten as

$$\dot{x}(ih)=-\frac{k}{f}x(ih)-\frac{k'}{f}x^3(ih) \quad \ldots (16a)$$

$$\ddot{x}(ih)=-\frac{k}{f}\dot{x}(ih)-\frac{3k'}{f}x^2(ih)\dot{x}(ih) \quad \ldots (16b)$$

So, knowing x(0), using (16a) and (16b), $\dot{x}(0)$ and $\ddot{x}(0)$ can be obtained.

Using (3), for the unknown function x(t), we can write the following equations as

$$\left.\begin{aligned}\bar{x}1(1) &= x(0)+h\,\dot{x}(0)\\ \bar{x}1\{(i+1)h\} &= \bar{x}1(ih)+h\,\dot{\bar{x}}1(ih)\end{aligned}\right\} \quad \ldots (17)$$

Thus, from (17), using (16a), we can solve for x(t) recursively using first order Taylor approximation.

Similarly, to obtain a more accurate solution, we use second order Taylor approximation as in (6) to get

$$\bar{x}2(h) = x(0)+h\,\dot{x}(0)+\frac{h^2}{2}\ddot{x}(0)$$

$$\bar{x}2\{(i+1)h\} = x(ih)+h\,\dot{x}(ih)+\frac{h^2}{2!}\ddot{x}(ih) \quad \ldots (18)$$

Thus, from (18), using both (16a) and (16b), we can solve for x(t) recursively using second order approximation. Also, equations (17) and (18) may be solved with a fixed value of the step h or a dynamic value. That is, h can be changed during recursion.

B. Example [5]

Consider nonlinear time invariant system

$$\dot{x} = -x - x^2 + 0.2 \qquad \text{where, } x(0) = 0 \quad \ldots (19)$$

From above equation we find,

$$\ddot{x} = -\dot{x} - 2x\dot{x} \quad \ldots (20)$$

Then using (19) and (20), we find solution at any point of time in any interval by using the following recursion

$$x\{(i+1)h\} = x(ih)+h\,\dot{x}(ih)+\frac{h^2}{2!}\ddot{x}(ih)$$

$$= x(ih)+h\,\dot{x}(ih)+\frac{h^2}{2!}\left(-\dot{x}(ih)-2x(ih)\dot{x}(ih)\right)$$

The exact solution is obtained using Volterra integral equation as

$$x(t) = \{-5.1095\exp(1.3416t) - 0.7453\}^{-1} + 0.1708$$

For m=20, T=4, the discrete solution points are presented in Table 3.
Solution by 1st order Taylor series is represented by Taylor1 and 2nd order Taylor series by Taylor2 in graph.

Figure 5 shows the recursive solutions obtained via first order Taylor approximation while figure 6 shows the solutions obtained using second order approximation, with m=20 in each case.

Table 3: Recursive solution of the state obtained via first order and second order Taylor approximation compared with the exact solution (for T=4 s, m=20 and h=0.2 s).

Time (Sec.)	Exact solution x(t)	Pointwise recursive solution of x(t) with 1st order approximation	Pointwise recursive solution of x(t) with 2nd order approximation
0	0	0	0
0.2000	0.0362	0.0400	0.0360
0.4000	0.0654	0.0717	0.0651
0.6000	0.0887	0.0963	0.0882
0.8000	0.1071	0.1152	0.1066
1.0000	0.1215	0.1295	0.1210
1.2000	0.1328	0.1402	0.1323
1.4000	0.1416	0.1483	0.1411
1.6000	0.1483	0.1542	0.1479
1.8000	0.1536	0.1586	0.1531
2.0000	0.1576	0.1619	0.1572
2.2000	0.1607	0.1642	0.1603
2.4000	0.1630	0.1660	0.1628
2.6000	0.1649	0.1673	0.1646
2.8000	0.1663	0.1682	0.1661
3.0000	0.1673	0.1689	0.1672
3.2000	0.1682	0.1694	0.1680
3.4000	0.1688	0.1698	0.1687
3.6000	0.1693	0.1701	0.1692
3.8000	0.1696	0.1703	0.1695

To compare the validity of these results, both the figures show the exact solution. As expected, the second order approximation is much better than the first order approximation. In figure 7, all the results, i. e., recursive solution using first order as well as second order Taylor approximation, and the exact solution, are shown together for better clarity.

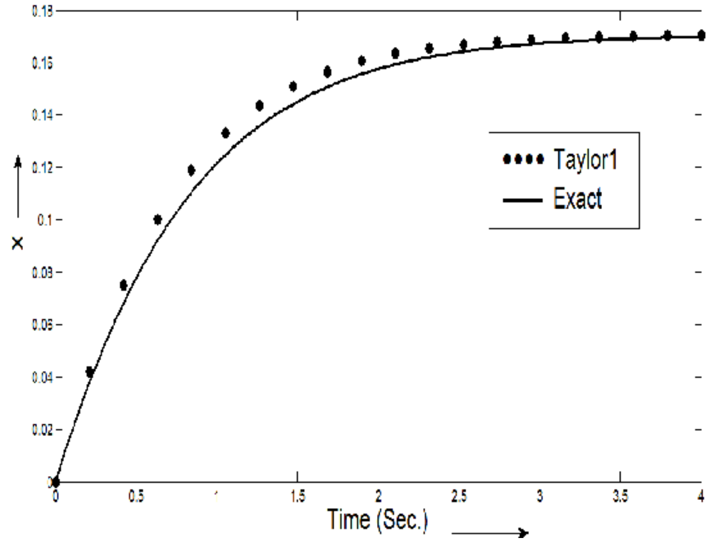

Fig. 5. First order Taylor approximation for m=20.

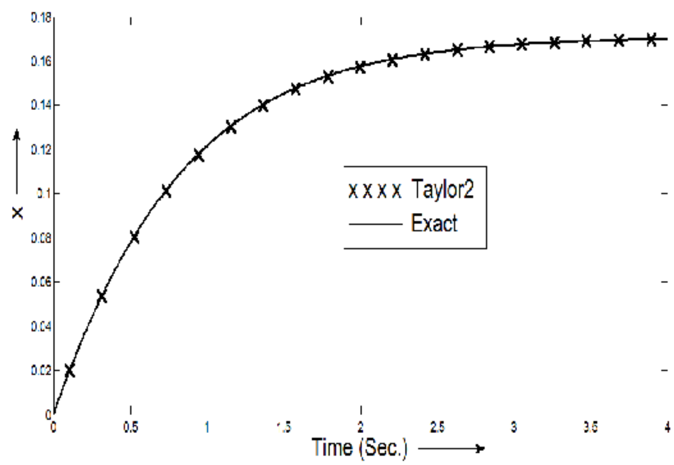

Fig. 6. Second order Taylor approximation for m=20.

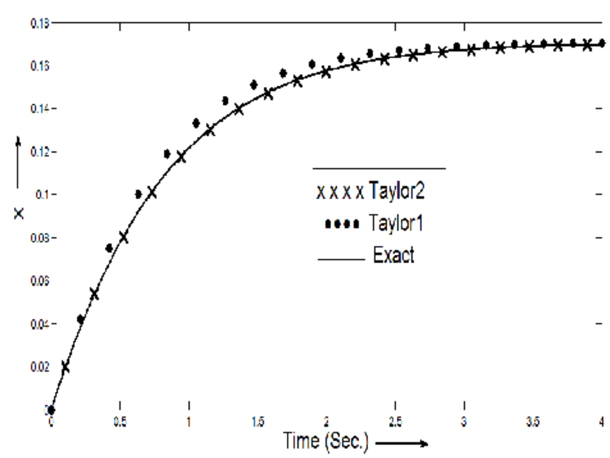

Fig. 7. Comparison between first order approximation and second order approximation for m = 20.

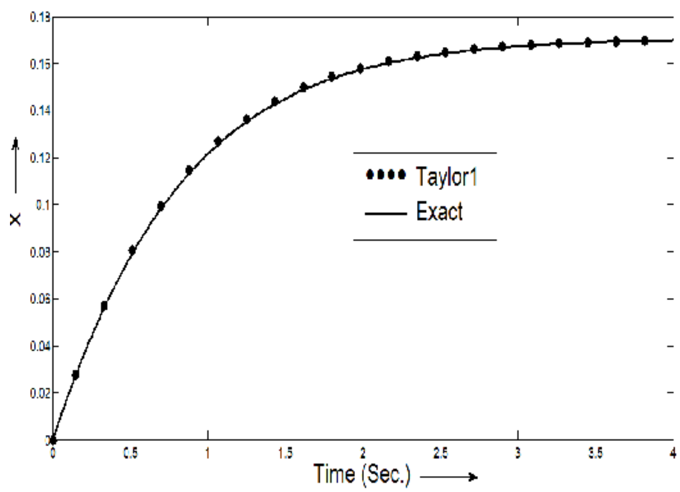

Fig. 8. First order Taylor approximation for m = 110.

In Figure 3 and 6, 1st order solution and 2nd order solution are solved with the same sub- interval h, but for the sake of clarity, the graph for Taylor 2 solution is plotted with an initial sampling period of h/2, and after that the plot is continued with the same sampling period h, so that the sample points do not mingle.
In Figure 8, no of intervals taken is 110. But for the sake of clarity, all of 110 points are not plotted. Instead, only a few points are indicated in the graph.

IV. CONCLUSION

We have presented recursive method for solving linear as well as nonlinear state equations of a control system based upon Taylor series expansion. We have used both first order and second order expansion using different step sizes. It is observed that the results obtained in this manner are reasonably reliable. For first order approximation, using 20 steps (m=20), the deviation of the solution from the exact results are quite noticeable. But for second order Taylor approximation the presented recursive method offers highly dependable solutions and almost overlaps with the exact solution. Three tables (tables 1, 2 and 3) and eight figures (fig. 1 to fig. 8) are presented to compare the results in both quantitative and qualitative manner.

It is clear from the plots of different figures, how close the Taylor series based solutions are. In fact, solutions obtained via Taylor 2 and also, with Taylor 1 (with smaller subinterval h of course), are very close to the exact solutions. Three tables (Tables 1, 2 and 3) also reflect this fact.

Solution of differential equation using Taylor series is not new. But the innovative idea in this work is to analyze a dynamic system by solving state equations using Taylor series, especially in a recursive manner. A recursive method is considered to be faster than methods involving Kronecker products and inversion of large matrices. And also, the recursive method requires much less memory. The simplicity of solution using Taylor series via recursion makes it powerful tool over other methods.

Also, solving nonlinear system is a more difficult task compared to solving linear systems. And it always calls for special techniques involving much more mathematical complexity. In this paper we have used the same tool to solve linear as well as nonlinear systems. We think this is really a specialty of this work.

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