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Convergence of *Abbas and Nazir* Iterates for a Multi-valued Map with a Fixed Point

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ABSTRACT: Defining *Abbas and Nazir* iteration for a Multi-valued mapping of *T* with an invariant point σ is the object of this paper along with explaining that under certain conditions, this iteration gets converged to an invariant point ω belonging to *T*. However, it is essential, to note that this invariant point ω is different from σ.

Keywords*:* Multi-valued Map, *Abbas and Nazir* iteration, Invariant points.

AMS Subject Classification: 47H10, 54H25

I. INTRODUCTION

Assuming (X, d) to be a complete metric space and X, having a subset K known to be $proximinal$, wherein, there is in existence, an element $k \in K$ for every $x \in X$, as

$$d(x, k) = d(x, K) = \inf\{d(x, y): y \in K\}$$

Every closed convex subset X has to be $proximinal$, for X being a Hilbert space. The families of all bounded $proximinal$ subsets of K in X, and those of nonempty bounded and closed subsets of X are denoted by $P(K)$ and $CB(X)$ respectively.

Assuming X having two bounded subsets namely A and B, the Hausdorff distance between them is defined as:

$$H(A, B) = max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(A, y)\right\}$$

There shall be a reference made to carve out a detailed analysis and review of literature with respect to *Abbas and Nazir iterates* by taking recourse to the *Abbas and Nazir* [1]. Transformation from single valued map to multi valued map, thereby extending the convergence results of single valued mapping with the aid of *Abbas and Nazir iteration* scheme shall be the focal point of this paper.

The Picard iteration sequence [7] for every $x_1 \in K$, defined as

$$x_{n+1} = f^n x, n \in \mathbb{N}$$

does not require to be converged with reference to nonexpansive mapping. The iteration sequence $x_{n+1} = f^n x$ which maps $f: [-1,1] \rightarrow [-1,1]$ and is defined by $fx = -x$ is not convergent to 0 for every non initial point (being non zero) which is, as a matter of fact, the invariant point of f. Mann [4] introduced an iteration scheme for non expansive mapping which was convergent iteration sequence for arbitrary $x_1 \in K$ as follows:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n f x_n, n \in \mathbb{N}$$

where $\alpha_n \in (0,1)$.

In any of the Hilbert spaces, Ishikawa's [3] introduction of new iteration process in 1974, for the approximation of the invariant point of pseudo-contractive compact mapping, is as follows:

for $x_1 \in K$

$$\begin{cases} y_n = (1 - \alpha_n)x_n + \alpha_n f x_n, \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n f y_n, \quad n \in \mathbb{N} \end{cases}$$

where $\alpha_n, \beta_n \in (0,1)$.

In order to compare two iteration schemes in one dimensional the scholar has referred Rhoades [10]. Herein, Ishikawa Iteration convergence rate is shown to better even that of Mann's Iteration procedure under favorable conditions. Nadler [9] and Markin [5] studied invariant points for Multi-valued non expansive mappings and it is for their efforts that now, there is an extensive and vast literature on Multi-valued invariant point theory having wide range of applications in diverse areas, be it optimization, or be it differential inclusion [6]. It is because of Lim [15] that the existence of invariant points belonging to mappings which are Multi-valued nonexpansive, in Banach spaces (characteristically uniformly convex), could be proved.

In order to approximate the invariant points of Multi-valued nonexpansive mappings, a number of iteration schemes processes have been used in the last few years. Among these, noteworthy generalizations of iteration processes given by Mann and Ishikawa, notably in cases of Multi-valued mapping can be seen in the iteration processes of Song and Wang [13], Sastry and Babu [11], Shahzad and Zegeye [12] and Panyanak [6].

It is not been long that a single valued iterate scheme was introduced by Abbas and Nazir [1] which provided for an iteration convergence rate, which was faster than that developed by Agarwal *et al.*, [2] which itself was faster than the one introduced earlier by Picard. The aforementioned iteration scheme is as follows:

:

$$\begin{cases} x_1 \in X, \\ x_{n+1} = (1-\alpha_n)Ty_n + \alpha_n Tz_n\,, \\ y_n = (1-\beta_n)Tx_n + \beta_n Tz_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n Tx_n\,, \qquad n \in \mathbb{N} \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$,$\{\gamma_n\}$ are real sequences satisfying $0 < \alpha_n,\ \beta_n,\ \gamma_n < 1$.

Different spaces having different mappings have been the subjects of various studies undertaken by several reputed authors [10-15] as a part of the schemes followed by them.

As, the iteration scheme is for single valued mapping, we will be introducing the same iteration scheme for Multi-valued mapping in this paper. X shall be taken to be the real Hilbert space for the remaining parts of this paper. We are introducing the following iteration scheme as:

Let us take mapping T defined from X to $P(X)$ and consider σ as an invariant point belonging to T. The *Abbas and Nazir* iteration sequence is defined as

$$\begin{cases} x_1 \in X \\ x_{n+1} = (1-\alpha_n)z'_n + \alpha_n z''_n\,, \text{where } z'_n \in Ty_n \\ \text{such that } ||z' - \sigma|| = d(Ty_n,\sigma), \\ z''_n \in Tz_n \ \textit{such}\ ||z' - \sigma|| = d(Ty_n,\sigma), \\ y_n = (1-\beta_n)z'''_n + \beta_n z''_n \text{ and} \\ z_n = (1-\gamma_n)x_n + \gamma_n z'''_n\,, \\ \text{where } z'''_n \in Tx_n \\ \text{such that } ||z'''_n - \sigma|| = d(Tx_n,\sigma), n \in \mathbb{N} \end{cases} \tag{A}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences satisfying $0 \le \alpha_n, \beta_n, \gamma_n < 1, \beta_n \to 0$ and $\sum \alpha_n\beta_n = \sum \gamma_n\beta_n = \sum \alpha_n\gamma_n = \infty$.

Preliminaries: The proof of theorems is studied by us using lemma and definitions as well as various results and iteration processes to make this presentation more closed and self contained.

Definitions. A mapping T satisfying different inequality shall have different definitions according to the satisfaction thereby achieved. Hence, the mapping is known as

– *Multi-valued nonexpansive* if

$H(Tx,Ty) \le ||x-y||$ for all $x,y \in K$.

– *Multi-valued generalized nonexpansive* if

$$H(Tx,Ty) \le \alpha||x-y|| + \beta\{d(x,Tx) + d(y,Ty)\} + \gamma\{d(x,Ty) + d(y,Tx)\}$$

for all $x,y \in X$ where $\alpha + 2\beta + 2\gamma \le 1$.

–*Multi-valued quasi-contractive* if for some $0 \le k < 1$, $H(Tx,Ty) \le max\{||x-y||, d(x,Tx), d(y,Ty) + d(x,Ty), d(y,Tx)\}$ for all $x,y \in X$.

The following lemmas will be useful in our subsequent discussion and are easy to establish.

Lemma 1. If $\{\alpha_n\}$, $\{\beta_n\}$ be two real sequences such that

(i) $0 \le \alpha_n,\beta_n < 1$,

(ii) $\beta_n \to 0$ as $n \to \infty$ and

(iii) $\sum \alpha_n\beta_n = \infty$.

If there is a real sequence $\{\gamma_n\} \in [0,\infty)$ existing in such a manner that $\sum \alpha_n\beta_n(1-\beta_n)\gamma_n$ having being bound, then γ_n has a subsequence which gets converged to 0.

Lemma 2. [8] .If $\{x_n\}$be a sequence of reals which satisfies $x_{n+1} \le \alpha_n x_n + \beta_n$ where $x_n \ge 0, \beta_n \ge 0$ and$\lim_{n\to\infty}\beta_n = 0$, $0 \le \alpha < 1$,then $\lim_{n\to\infty} x_n = 0$.

Lemma 3. [3] If $\theta \in [0,1]$, then for any $x,y \in X$,

$$||(1-\theta)x + \theta y||^2 = (1-\theta)||x||^2 + \theta||y||^2 - \theta(1-\theta)||x-y||^2.$$

II. MAIN RESULTS

Theorem 4. Suppose that there is a Hilbert space X having a subset K which is compact and convex and also that there is a non expansive mapping T defined from K to $P(K)$ has an invariant point σ. Having assumed that,

(i) α_n,$\beta_n \in [0,1)$

(ii) $\beta_n \to 0$ and

(iii) $\sum \alpha_n\beta_n = \infty$.

Then, the *Abbas and Nazir* iteration sequence characterized by (A) gets converged to an invariant point ω belonging to T.

$Proof$. Now, if we use lemma 2,

$$\begin{aligned} ||x_{n+1} - \sigma||^2 &= ||(1-\alpha_n)z'_n + \alpha_n z'' - \sigma||^2 \\ &= (1-\alpha_n)||z'_n - \sigma||^2 + \alpha_n||z''_n - \sigma||^2 - \alpha_n(1-\alpha_n)||z'_n - z''_n||^2 \\ &\le (1-\alpha_n)H^2(Ty_n,T\sigma) + \alpha_n H^2(Tz_n,T\sigma) - \alpha_n(1-\alpha_n)||z'_n - z''_n||^2 \\ &\le (1-\alpha_n)||y_n - \sigma||^2 + \alpha_n||z_n - \sigma||^2 - \alpha_n(1-\alpha_n)||z'_n - z''_n||^2 \end{aligned} \tag{1}$$

$$\begin{aligned} ||y_n - \sigma||^2 &= ||(1-\beta_n)z'''_n + \beta_n z''_n - \sigma||^2 \\ &= (1-\beta_n)||z'''_n - \sigma||^2 + \beta_n||z''_n - \sigma||^2 - \beta_n(1-\beta_n)||z'''_n - z''_n||^2 \\ &\le (1-\beta_n)H^2(Tx_n,T\sigma) + \alpha_n H^2(Tz_n,T\sigma) - \beta_n(1-\beta_n)||z'''_n - z''_n||^2 \\ &\le (1-\beta_n)||x_n - \sigma||^2 + \beta_n||z_n - \sigma||^2 - \beta_n(1-\beta_n)||z'''_n - z''_n||^2 \end{aligned} \tag{2}$$

$$\begin{aligned} ||z_n - \sigma||^2 &= ||(1-\gamma_n)x_n + \gamma_n z'''_n - \sigma||^2 \\ &= (1-\gamma_n)||x_n - \sigma||^2 + \gamma_n||z'''_n - \sigma||^2 - \gamma_n(1-\gamma_n)||x_n - z'''_n||^2 \\ &\le (1-\gamma_n)||x_n - \sigma||^2 + \gamma_n H^2(Tx_n,T\sigma) - \gamma_n(1-\gamma_n)||x_n - z'''_n||^2 \\ &\le (1-\gamma_n)||x_n - \sigma||^2 + \gamma_n||x_n - \sigma||^2 - \gamma_n(1-\gamma_n)||x_n - z'''_n||^2 \\ &\le ||x_n - \sigma||^2 - \gamma_n(1-\gamma_n)||x_n - z'''_n||^2 \end{aligned} \tag{3}$$

Now, we substitute (3) in (2)

$$\begin{aligned} ||y_n - \sigma||^2 &\le (1-\beta_n)||x_n - \sigma||^2 + \beta_n[\ ||x_n - \sigma||^2 - \gamma_n(1-\gamma_n)||x_n - z'''_n||^2] - \beta_n(1-\beta_n)||z'''_n - z''_n||^2 \\ &\le ||x_n - \sigma||^2 - \beta_n\gamma_n(1-\gamma_n)||x_n - z'''_n||^2 - \beta_n(1-\beta_n)||z'''_n - z''_n||^2 \end{aligned} \tag{4}$$

Now, we substitute (3) and (4) in (1)

$$\begin{aligned} ||x_{n+1} - \sigma||^2 &\le (1-\alpha_n)||y_n - \sigma||^2 + \alpha_n||z_n - \sigma||^2 - \alpha_n(1-\alpha_n)||z'_n - z''_n||^2 \\ &\le (1-\alpha_n)[||x_n - \sigma||^2 - \beta_n\gamma_n(1-\gamma_n)||x_n - z'''_n||^2 - \beta_n(1-\beta_n)||z'''_n - z''_n||^2\,] \\ &\quad + \alpha_n[||x_n - \sigma||^2 - \gamma_n(1-\gamma_n)||x_n - z'''_n||^2\ ] - \alpha_n(1-\alpha_n)||z'_n - z''_n||^2 \\ &\le ||x_n - \sigma||^2 - [\beta_n\gamma_n(1-\alpha_n)(1-\gamma_n) - \alpha_n\gamma_n(1-\gamma_n)]||x_n - z'''_n||^2 \\ &\quad - \beta_n(1-\beta_n)(1-\alpha_n)||z'''_n - z''_n||^2 - \alpha_n(1-\alpha_n)||z'_n - z''_n||^2 \end{aligned}$$

Therefore,

$$[\beta_n\gamma_n(1-\alpha_n)(1-\gamma_n)-\alpha_n\gamma_n(1-\gamma_n)]\left||x_n - z'''\right||^2 + \beta_n(1-\beta_n)(1-\alpha_n)\left||z'''-z''\right||^2 + \alpha_n(1-\alpha_n)||z_n' - z''||^2 \le \left||x_n-\sigma\right||^2 + ||x_{n+1}-\sigma||^2$$

Therefore

$$\sum[\beta_n\gamma_n(1-\alpha_n)(1-\gamma_n)-\alpha_n\gamma_n(1-\gamma_n)]\left||x_n - z_n'''\right||^2 \le \left||x_1-\sigma\right|| < \infty.$$

By lemma 1

There is a subsequence $\{x_{n_l} - z_{n_l}'''\}$ of $\{x_n - z_n'''\}$ existing, so much, so that $||x_{n_l} - z_{n_l}'''|| \to 0$ as $l \to \infty$. Since, $z_{n_l}''' \in Tx_{n_l}$,

$d(Tx_{n_l},\ x_{n_l}) \le ||x_{n_l} - z_{n_l}'''|| \to 0$

because $||x_{n_l} - z_{n_l}'''|| \to 0$ when $l \to \infty$ with $\{x_{n_l}\} \subset K$, $as\, K$ is complete, without having lost any generality. Now assuming that $x_{n_l} \to \omega$ as $l \to \infty$, $d(Tx_{n_l}, \omega) \le d(Tx_{n_l},\ x_{n_l}) + ||x_{n_l} - \omega|| \to 0$ as $l \to \infty$. Also $H(\ d(Tx_{n_l}, T\omega)) \to 0$ as $l \to \infty$.

Hence

$d(T\omega,\omega) \le d(\omega,\ Tx_{n_l}) + H(Tx_{n_l},T\omega) \to 0\ as\ l \to \infty.$

This shows that $\omega \in T\omega$. The theorem is followed thereby.

Theorem 5. Suppose that while X being a Hilbert space having K as *a* subset which is compact and convex, the generalized nonexpansive mapping T defined from K to $P(K)$ having an invariant point σ, let's assume

(i) $\alpha_n,\beta_n \in [0,1)$

(ii) $\beta_n \to 0$ and

$(iii) \sum \alpha_n \beta_n = \infty$.

Then, the *Abbas and Nazir* iteration sequence characterized by (A) gets converged to an invariant point ω belonging to T.

Proof. Having,

$$\left||x_{n+1}-\sigma\right||^2 \le (1-\alpha_n)H^2(Ty_n,T\sigma) + \alpha_n H^2(Tz_n,T\sigma) - \alpha_n(1-\alpha_n)||z_n' - z_n''||^2 \quad (5)$$

and with T, having the generalized nonexpansive characteristic, we get

$$\begin{aligned} H(T\sigma,Ty_n) &\le a\left||y_n-\sigma\right|| + b\, d(y_n,Ty_n) + c\{d(\sigma,Ty_n)+d(y_n,T\sigma)\} \\ &\le a\left||y_n-\sigma\right|| + b\{\left||y_n-\sigma\right|| + d(\sigma,Ty_n)\} + c\{d(\sigma,Ty_n)+d(y_n.T\sigma)\} \\ &\le (a+b+c)\left||y_n-\sigma\right|| + (b+c)d(\sigma,Ty_n) \\ &\le (a+b+c)\left||y_n-\sigma\right|| + (b+c)\,H(T\sigma,Ty_n) \end{aligned}$$

Hence

$$H(T\sigma,Ty_n) \le \frac{a+b+c}{1-(b+c)}\left||y_n-\sigma\right|| \quad (6)$$

Since $\frac{a+b+c}{1-(b+c)} \le 1$, it follows that

$$H(T\sigma,Ty_n) \le ||y_n-\sigma||$$

from (5) and (6), we have

$$\left||x_{n+1}-\sigma\right||^2 \le (1-\alpha_n)\left||y_n-\sigma\right||^2 + \alpha_n\left||z_n-\sigma\right||^2 - \alpha_n(1-\alpha_n)||z_n'-z_n''||^2$$

which is the inequality (1).

In the same way, it is of very little significance showing the inequality (2) and (3) holding as

$$\left||y_n-\sigma\right||^2 \le (1-\beta_n)\left||x_n-\sigma\right||^2 + \beta_n\left||z_n-\sigma\right||^2 - \beta_n(1-\beta_n)||z_n'''-z_n''||^2$$

and

$$||z_n-\sigma||^2 \le ||x_n-\sigma||^2 - \gamma_n(1-\gamma_n)||x_n-z_n'''||^2$$

Now, proceeding as we did with the proof of Theorem 4, the aforementioned theorem necessarily follows.

Theorem 6. Suppose, X is a Hilbert space having a subset K which is closed as well as convex and bounded, and that T is a mapping defined from K to $P(K)$ is a mapping and has an invariant point σ. Suppose real sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in such a manner, that

(i) $\alpha_n,\ \beta_n, \gamma_n \in [\,0{,}1\,)$ for all n

(ii) $\beta_n \to 0$ whenever $n \to \infty$ with

(iii) $\delta \le \alpha_n, \gamma_n \le 1-k^2$ for some positive real δ.

Thereby, Abbas and Nazir iteration sequence as is defined by (A), gets converged to σ of T.

Proof. By using lemma

$$||x_{n+1}-\sigma||^2 = ||z_n'-\sigma||^2 \quad (7)$$

$$||z_n'-\sigma||^2 = d(\sigma,Ty_n) \le H(T\sigma,Ty_n)$$

Therefore

$$\left||z_n'-\sigma\right||^2 \le H(T\sigma,Ty_n) \le k^2\, max_{z\in T\sigma}\{\left||y_n-\sigma\right||^2, d^2(y_n,Ty_n), d^2(\sigma,Ty_n)\} \quad (8)$$

Since, $d^2(y_n,Ty_n) \le \left||y_n-\sigma\right||^2$

If $d(\sigma,Ty_n)$ is the maximum, then

$$H^2(T\sigma,Ty_n) \le k^2 d^2(\sigma,Ty_n) \le H^2(T\sigma,Ty_n)$$

So that $0 \le ||z_n'-\sigma||^2 \le H^2(T\sigma,Ty_n) = 0$. Hence from (8) we get, always,

$$\begin{aligned} \left||z_n'-\sigma\right||^2 &\le H(T\sigma,Ty_n) \\ &\le k^2\, max\,\{\left||y_n-\sigma\right||^2, d^2(y_n,Ty_n)\} \\ &\le k^2\ [\left||y_n-\sigma\right||^2 + d^2(y_n,Ty_n)] \end{aligned} \quad (9)$$

On the other hand,

$$||z_n''-\sigma||^2 = d(\sigma,Tz_n) \le \max_{y\in Tp} d(y,Tz_n) \le H(T\sigma,Tz_n)$$

Therefore

$$\left||z_n'-\sigma\right||^2 \le H(T\sigma,Tz_n) \le k^2 \max_{z\in Tp}\{\left||z_n-\sigma\right||^2, d^2(z_n,Tz_n), d^2(\sigma,Tz_n)\} \quad (10)$$

(Since, $d^2(z_n,Tz_n) \le \left||z_n-\sigma\right||^2$)

If $d(\sigma,Tz_n)$ is the maximum, then

$$\begin{aligned} H^2(T\sigma,Tz_n) &\le k^2 d^2(\sigma,Tz_n) \\ &\le k^2 \max_{y\in Tp}\ d^2(y,Tz_n) \\ &\le k^2 H^2(T\sigma,Tz_n) \end{aligned}$$

So that $0 \le ||z_n''-\sigma||^2 \le H^2(T\sigma,Tz_n) = 0$. Hence from (10) we get, always,

$$\begin{aligned} \left||z''-\sigma\right||^2 &\le H(T\sigma,Tz_n) \\ &\le k^2\, max\,\{\left||z_n-\sigma\right||^2, d^2(z_n,Tz_n)\} \\ &\le k^2\ [\left||z_n-\sigma\right||^2 + d^2(z_n,Tz_n)] \end{aligned} \quad (11)$$

Similarly,

$$\begin{aligned} \left||z_n'''-\sigma\right||^2 &\le H(T\sigma,Tx_n) \\ &\le k^2\, max\,\{\left||x_n-\sigma\right||^2, d^2(x_n,Tx_n)\} \\ &\le k^2\ [\left||x_n-\sigma\right||^2 + d^2(x_n,Tx_n)] \end{aligned} \quad (12)$$

Now consider

$$\begin{aligned} \left||y_n-\sigma\right||^2 &= ||(1-\beta_n)z_n''' + \beta_n z_n'' - \sigma||^2 \\ &= (1-\beta_n)\left||z_n'''-\sigma\right||^2 + \beta_n\left||z_n''-\sigma\right||^2 \\ &\quad -\beta_n(1-\beta_n)||z_n'''-z_n''||^2 \end{aligned} \quad (13)$$

$$\begin{aligned} d^2(y_n,Ty_n) &\le \left||y_n - z_n'\right||^2 \\ &= ||(1-\beta_n)z_n''' + \beta_n z_n'' - z_n'||^2 \end{aligned}$$

$$= (1-\beta_n)\|z_n''' - z_n'\|^2 + \beta_n\|z_n'' - z_n'\|^2 - \beta_n(1-\beta_n)\|z_n''' - z_n''\|^2 \quad (14)$$

Also,

$$\|z_n - \sigma\|^2 = \|(1-\gamma_n)x_n + \gamma_n z_n''' - \sigma\|^2 = (1-\gamma_n)\|x_n - \sigma\|^2 + \gamma_n\|z_n''' - \sigma\|^2 - \gamma_n(1-\gamma_n)\|x_n - z_n'''\|^2 \quad (15)$$

$$d^2(z_n, Tz_n) \le \|z_n - z_n''\|^2 = \|(1-\gamma_n)x_n + \gamma_n z_n''' - z_n''\|^2 = (1-\gamma_n)\|x_n - z_n''\|^2 + \gamma_n\|z_n''' - z_n''\|^2 - \gamma_n(1-\gamma_n)\|x_n - z_n'''\|^2 \quad (16)$$

Now, we substitute (13) and (14) in (9)

$$\|z_n' - \sigma\|^2 \le k^2 \Big[(1-\beta_n)\|z_n''' - \sigma\|^2 + \beta_n\|z_n'' - \sigma\|^2 - 2\beta_n(1-\beta_n)\|z_n''' - z_n''\|^2 + (1-\beta_n)\|z_n''' - z_n'\|^2 + \beta_n\|z_n'' - z_n'\|^2\Big] \quad (17)$$

Similarly, if we substitute (15) and (16) in (11), we have

$$\|z_n'' - \sigma\|^2 \le k^2\Big[(1-\gamma_n)\|x_n - \sigma\|^2 + \gamma_n\|z_n''' - \sigma\|^2 - 2\gamma_n(1-\gamma_n)\|x_n - z_n'''\|^2 + (1-\gamma_n)\|x_n - z_n''\|^2 + \gamma_n\|z_n''' - z_n''\|^2\Big] \quad (18)$$

From (12), (17) and (18)

$$\|z_n'' - \sigma\|^2 \le [(1-\beta_n)k^4 + \beta_n k^2 + \beta_n\gamma_n k^4]\|x_n - \sigma\|^2 - 2\beta_n\gamma_n k^2\|x_n - z_n'''\|^2 + [\beta_n k^2\gamma_n - 2\beta_n k^2(1-\beta_n)]\|z_n''' - z_n''\|^2 + k^2\beta_n\|z_n'' - z_n'\|^2 \,] + [(1-\beta_n)k^4 + \beta_n\gamma_n k^4]\, d^2(x_n, Tx_n) \quad (19)$$

Now, we substitute (18) in (7)

$$\|x_{n+1} - \sigma\|^2 \le (1-\alpha_n)[((1-\beta_n)k^4 + \beta_n k^2 + \beta_n\gamma_n k^4 + \alpha_n]\|x_n - \sigma\|^2 - 2\beta_n k^2\gamma_n\|x_n - z_n'''\|^2 + \beta_n k^2(1-\gamma_n)\|x_n - z_n''\|^2 + [(1-\beta_n) + \beta_n k^2\gamma_n - 2k^2\beta_n(1-\beta_n)]\|z_n''' - z_n''\|^2 + [\beta_n k^2 - \alpha_n(1-\alpha_n)]\|z_n'' - z_n'\|^2 + [(1-\beta_n)k^4 + \beta_n\gamma_n k^4 + \alpha_n]\, d^2(x_n, Tx_n)\,] \quad (20)$$

Since, there exists a positive integer N_1 such that $\beta_n = k^{\frac{1}{2}}$ also we have $\delta \le \gamma_n \le 1 - k^2$, we have $\beta_n k^2(1-\gamma_n) \le (1-\delta) = \gamma\ (say)$ and $0 < \gamma < 1$, $\gamma_n \to 0$ as $n \to \infty$ for all $n \ge N_1$. In a similar manner, there exists a positive integer N_2 such that

$\gamma_n \le (1 + k^{\frac{1}{2}})k^4$ and $\alpha_n \le k^4$ for all $n \ge N_2$

so that,

$(1-\alpha_n)[((1-\beta_n)k^4 + \beta_n k^2 + \beta_n\gamma_n k^4 + \alpha_n] \le (1 + k^4) = \alpha(\text{say})$

for all $n \ge N_2$.

Similarly, it is easy to show that

$$[(1-\beta_n)k^4 + \beta_n\gamma_n k^4 + \alpha_n] \ge 0.$$

for every n, being adequately large,

$$\|x_{n+1} - \sigma\|^2 \le \alpha\|x_{n+1} - \sigma\|^2 + [1 - \beta_n + \beta_n k^2\gamma_n - 2k^2\beta_n(1-\beta_n)\beta_n k^2 - \alpha_n(1-\alpha_n)]D$$

having D as the diameter measuring k, the convergence to σ of the sequence $\{x_n\}$ takes place when $n \to \infty$, thereby allowing the theorem to follow.

III. REMARK

A well illustrated example [12] proved that the limit of the sequence of *Abbas and Nazir* iterates depends on the choice of the invariant point ω and the initial choice of x_1 and the invariant point may be different from σ.

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