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Fixed Point Results in Parametric Metric Space

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ABSTRACT: In the present paper, we prove fixed point theorems based on rational expressions for contraction mapping in parametric space. Moreover, we provide an example to furnish our result and usability of our result.

Keywords: Contraction mapping, Fixed point, Parametric metric space, Complete metric space, Convergent.

I. INTRODUCTION

Let $A \neq \emptyset$ and $T: A \to A$ is a mapping, then a point $p^* \in A$ s.t $Tp^* = p^*$ is a fixed point of mapping T. If $T: A \to A$ is a multi-valued map (i.e. from $A \neq \emptyset$ is a subsets of A), then point $p^* \in A$ is afixed point of mapping T if $p^* \in Tp^*$. Most of the physical problems can be transferred to fixed point equation. Properly its origin goes back to the starting of 20th century as an important part of nonlinear study. Fixed point theory has mesmerized lots of researchers. In 1922, Banach (Polish mathematician) celebrated his most famous principle which is known as Banach contraction principle [1], to find a fixed point of a map. Only lagoon of Banach contraction principle was the mapping T must be continuous throughout space. A Banach contraction principle every popular among researchers and helpful in fixed point theory. Kannan [2], rectified the lagoon of Banach contraction principle and proved a fixed point theorem for operators that need not be continuous. Further, Chatterjea [3], proved a result for discontinuous mapping which is a kind of dual of Kannan mapping. A lucid survey shows that there exists a vast literature available on fixed point theory. Fixed point theorems are main concerned about existence and uniqueness of a point in a non empty set. They are applicable in iteration methods, partial differential equations, integral differential equations, variational inequalities etc. There are lots of authors who extended the Banach Contraction Principle in different directions. Since Banach contraction principle has seen many extension and generalization in different space [4-19].

In many are the concept of metric space is generalized. In last few years, different generalized metric space has been developed by different authors by changing triangular inequality using different approach. Some generalized metric space are D-metric, D^*metric space, b metric space, b-like metric space, partial metric space, partial b-metric space, quasi partial b metric space, Cone metric, Generalized cone metric space, etc.

Wang *et al.*, [25], introduced and defined the expansive mapping on complete metric space and proved some fixed point theorems. Moreover, Daffer and Kaneko [26] proved some fixed point results for couple of mappings on complete metric space using expansive mapping.

Recently, the idea of a parametric metric space gave by Hussian [27], in 2014 and proved some fixed point theorems on parametric metric space. Furthermore, information on parametric metric space we suggest the reader [28-30]. In the present paper, we prove fixed point theorems with contraction condition in parametric metric spaces.

II. PRELIMINARIES

Before going to the main results, here are some definitions, lemmas, properties and examples in sequel, most of are taken from the work of [27]. All the basic definitions will be useful for us to understand the work presented in the next section.

Definition 2.1: Let $A \neq \emptyset$ and let a function $T_p: A \times A \times (0, \infty) \to [0, \infty]$ then, T_p is called parametric metric space in A if

1. $T_p(f_*, g_*, t) = 0 \ \ iff \ \ f_* = g_*$
2. $T_p(f_*, g_*, t) = T_p(g_*, f_*, t)$
3. $T_p(f_*, g_*, t) \leq T_p(f_*, h_*, t) + T_p(h_*, g_*, t)$
   $for\ all\ \ f_*, g_*, h_* \in A\ and\ all\ t > 0.$

Pair (A, T_p) is called parametric metric space.

Example 2.2: Let $A = R^2$ for any $\alpha = (\alpha_1(t), \alpha_2(t))$, $\beta = (\beta_1(t), \beta_2(t))$ and define the function $T_p: A \times A \times (0, +\infty) \to [0, +\infty)$ by

$$T_p(\alpha, \beta, t) = |\alpha_1(t) - \beta_1(t)| + |\alpha_2(t) - \beta_2(t)|$$

$\forall\ \alpha, \beta \in A$ and all $t > 0$. Then T_p is parametric metric in A and (A, T_p) is parametric metric space.

Proof: For all $\alpha(t), \beta(t), \gamma(t) \in A$, we have

(i) $T_p(\alpha, \beta, t) = 0 \Rightarrow |\alpha_1(t) - \beta_1(t)| + |\alpha_2(t) - \beta_2(t)| = 0$

$$\Rightarrow |\alpha_1(t) - \beta_1(t)| = 0 \text{ and } |\alpha_2(t) - \beta_2(t)| = 0$$
$$\Rightarrow \alpha_1(t) - \beta_1(t) = 0 \text{ and } \alpha_2(t) - \beta_2(t) = 0$$
$$\Rightarrow \alpha_1(t) = \beta_1(t) \text{ and } \alpha_2(t) = \beta_2(t)$$

(ii) $T_p(\alpha, \beta, t) = |\alpha_1(t) - \beta_1(t)| + |\alpha_2(t) - \beta_2(t)|$

$$= |-(\beta_1(t) - \alpha_1(t))| + |-(\beta_2(t) - \alpha_2(t))|$$
$$= |\beta_1(t) - \alpha_1(t)| + |\beta_2(t) - \alpha_2(t)|$$

$$= T_p(\beta_1, \alpha_1, t) + T_p(\beta_2, \alpha_2, t)$$

(iii) $T_p(\alpha, \beta, t) = |\alpha_1(t) - \beta_1(t)| + |\alpha_2(t) - \beta_2(t)|$

$$\leq |\alpha_1(t) - \gamma_1(t)| + |\gamma_1(t) - \beta_1(t)| + |\alpha_2(t) - \gamma_2(t)| + |\gamma_2(t) - \beta_2(t)|$$

$$= T_p(\alpha,\gamma,t) + T_p(\gamma,\beta,t)$$

All the condition is satisfying the property of parametric metric space. Then, $T_p(\alpha,\beta,t)$ is called parametric metric space.

Definition 2.3: Consider $\{a_j\}$ be a sequence in parametric metric space (A,T_p) if

1. $\{a_j\}$ is known as convergent to $a \in A$ as, $\lim_{j\to\infty} a_j = 0, \forall \ t>0$ if $\lim_{j\to\infty} T_p(a_j, a, t) = 0$
2. $\{a_j\}$ is known as Cauchy sequence in A if $\forall \ t>0$, if $\lim_{j,i\to\infty} T_p(a_j, a_i, t) = 0$.
3. Every Cauchy sequence (A,T_p) is a convergent sequence then that sequence (A,T_p) is called complete.

Definition 2.4: Let (A,T_p) is a parametric metric space and a function $T: A \to A$ is continuous at $a \in A$, if for any sequence $\{a_j\}$ in A s.t

$$\lim_{j\to\infty} a_j = a \ \text{then} \lim_{j\to\infty} Ta_j = Ta$$

Lemma 2.5: Let construct a sequence $\{m_k\}$ in a parametric metric space(A,T_p) s.t

$$T_p(m_k, m_{k+1}, t) \le hT_p(m_{k-1}, m_k, t)$$

Where,$h \in [0,1)$ and $k = 1,2,3 \ldots\ldots$

Then $\{m_k\}$ is a Cauchy sequence in (A,T_p)

Verification: Let $k > l \ge 1$, it follows that

$$T_p(m_k, m_l, t) \le T_p(m_k, m_{k+1}, t) + T_p(m_{k+1}, m_{k+2}, t) \ldots + (m_{l-1}, m_l, t)$$

$$\le (h^k + h^{k+1} \ldots\ldots h^{k-1}) T_p(m_0, m_1, t)$$

$\forall \ t > 0$.Since $h < 1$. Assume that $T_p(m_0, m_1, t) > 0$. By taking $lim_{k,l\to+\infty}$, we get

$$\lim_{k,l\to+\infty} T_p(m_k, m_l, t) = 0$$

As a result, $\{m_k\}$ is a Cauchy sequence in A. Also if $T_p(m_0, m_1, t) = 0 \ then \ T_p(m_k, m_l, t) = 0 \ \forall \ k > l$

Hence $\{m_k\}$ is Cauchy sequence in A.

III. MAIN RESULT

The objective of this section is to prove some fixed point results for continuous function as well as satisfy the contraction conditions by considering the self-mapping on parametric metric space.

Theorem 3.1: Let (A,T_p) is a complete parametric metric space and mapping $T: A \to A$ is a continuous then it satisfied the below condition:

$$T_p(Ta, Tb, t) \le \alpha_1 T_p(a,b,t) + \alpha_2[T_p(a,Ta,t) + T_p(b,Tb,t)] + \alpha_3[T_p(a,Tb,t) + T_p(b,Ta,t)] + \alpha_4\left[\frac{T_p(a,b,t)T_p(a,Tb,t)}{T_p(a,b,t)+T_p(b,Tb,t)}\right] + \alpha_5\left[\frac{T_p(a,Tb,t)T_p(b,Tb,t)}{T_p(a,b,t)+T_p(b,Tb,t)}\right] \quad (1)$$

Where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \ge 0$ with$\alpha_1 + 2\alpha_2 + 4\alpha_3 + \alpha_4 + \alpha_5 < 1$ for all $a, b \in A$. and $t > 0$. Then T has a unique fixed point.

Proof: Let m_0 be an initial point and $\{m_j\}$ is a sequence such that $m_j = Tm_{j-1} = T^j m_0$. If there is a point $m_0 \in A$ such that $m_j = m_{j+1}$, then m_j is a fixed point. Therefore there is no need to precede further, Otherwise $m_j \ne m_{j+1}$. Using the inequality (1), we have

$$T_p(a,b,t) = T_p(Tm_j, Tm_{j+1}, t) \le \alpha_1 T_p(m_j, m_{j+1}, t)$$
$$+\alpha_2[T_p(m_j, Tm_j, t) + T_p(m_{j+1}, Tm_{j+1}, t)]$$
$$+\alpha_3[T_p(m_j, Tm_{j+1}, t) + T_p(m_{j+1}, Tm_j, t)]$$
$$+\alpha_4\left[\frac{T_p(m_j, m_{j+1}, t).T_p(m_j, Tm_{j+1}, t)}{T_p(m_j, m_{j+1}, t) + T_p(m_{j+1}, Tm_{j+1}, t)}\right]$$
$$+\alpha_5\left[\frac{T_p(m_j, Tm_{j+1}, t).T_p(m_{j+1}, Tm_{j+1}, t)}{T_p(m_j, m_{j+1}, t) + T_p(m_{j+1}, Tm_{j+1}, t)}\right]$$
$$\le \alpha_1 T_p(m_j, m_{j+1}, t)$$
$$+\alpha_2[T_p(m_j, m_{j+1}, t) + T_p(m_{j+1}, m_{j+2}, t)]$$
$$+\alpha_3[T_p(m_j, m_{j+2}, t) + T_p(m_{j+1}, m_{j+1}, t)]$$
$$+\alpha_4\left[\frac{T_p(m_j, m_{j+1}, t)T_p(m_j, m_{j+2}, t)}{T_p(m_j, m_{j+1}, t) + T_p(m_{j+1}, m_{j+2}, t)}\right]$$
$$+\alpha_5\frac{T_p(m_j, m_{j+2}, t)T_p(m_{j+1}, m_{j+2}, t)}{T_p(m_j, m_{j+1}, t) + T_p(m_{j+1}, m_{j+2}, t)}$$
$$\le \alpha_1 T_p(m_j, m_{j+1}, t)$$
$$+\alpha_2[T_p(m_j, m_{j+1}, t) + T_p(m_{j+1}, m_{j+2}, t)]$$
$$+2\alpha_3[T_p(m_j, m_{j+1}, t) + T_p(m_{j+1}, m_{j+2}, t)]$$
$$+\alpha_4 T_p(m_j, m_{j+1}, t) + \alpha_5 T_p(m_{j+1}, m_{j+2}, t)$$
$$T_p(m_{j+1}, m_{j+2}, t) \le \frac{\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4}{1 - (\alpha_2 + 2\alpha_3 + \alpha_5)} T_p(m_j, m_{j+1}, t)$$

Let $h = \frac{\alpha_1+\alpha_2+2\alpha_3+\alpha_4}{1-(\alpha_2+2\alpha_3+\alpha_5)}$ where $h = \alpha_1 + 2\alpha_2 + 4\alpha_3 + \alpha_4 + \alpha_5 < 1$.Therefore

$$T_p(m_{j+1}, m_{j+2}, t) \le hT_p(m_j, m_{j+2}, t)$$

Similarly

$$T_p(m_j, m_{j+1}, t) \le hT_p(m_{j-1}, m_j, t) \le h^2 T_p(m_{j-1}, m_j, t)$$

Using iteration up to j times,

$$T_p(m_j, m_{j+1}, t) \le h^j T_p(m_0, m_1, t)$$

where, $0 \le h \le 1$ and t >0 $\Rightarrow h^j \to 0$ as $j \to \infty$.Using lemma 2.5 sequence $\{m_j\}$ is Cauchy sequence. So $\exists$ μj such that $m_j \to \mu$ as $j \to \infty$. Next, we will show that μ is a fixed point of T. For that $m_j \to \mu$ as $j \to \infty$.By means of continuity into T, we have

$$\lim_{j\to\infty} Tm = T\mu$$
$$\lim_{j\to\infty} m_{j+1} = T\mu$$

Then, $T\mu = \mu$, then μ is a fixed point of T.

For uniqueness, let μ and ρ be the two fixed point of Tfor $\mu \ne \rho$,we have

$$T_p(\mu,\rho,t) \le \alpha_1 T_p(\mu,\rho,t) + \alpha_2[T_p(\mu,T\rho,t) + T_p(\rho,T\rho,t)]$$
$$+\alpha_3[T_p(\mu,T\rho,t) + T_p(\rho,T\mu,t)] + \alpha_4\left[\frac{T_p(\mu,\rho,t).T_p(\mu,T\rho,t)}{T_p(\mu,\rho,t)+T_p(\rho,T\rho,t)}\right]$$
$$+\alpha_5\frac{T_p(\mu,T\rho,t).T_p(\rho,T\rho,t)}{T_p(\mu,\rho,t)+T_p(\rho,T\rho,t)}$$

As μ and ρ are fixed point of T.

Therefore, by above equation we have,

$$T_p(\mu,\mu,t) = 0 \text{ and } T_p(\rho,\rho,t) = 0$$

So, above equation become

$$T_p(\mu,\rho,t) \le [\alpha_1 + \alpha_3 + \alpha_4]T_p(\mu,\rho,t) + \alpha_3 T_p(\rho,\mu,t) \quad (2)$$

Similarly

$$T_p(\rho,\mu,t) \le [\alpha_1 + \alpha_3 + \alpha_4]T_p(\rho,\mu,t) + \alpha_3 T_p(\mu,\rho,t) \quad (3)$$

Subtract (2) from (1)

$$|T_p(\mu,\rho,t) - T_p(\rho,\mu,t)|$$
$$\le |(\alpha_1 + \alpha_3 + \alpha_4) - \alpha_3||T_p(\mu,\rho,t) - T_p(\rho,\mu,t)|$$
$$\le |(\alpha_1 + \alpha_3 + \alpha_4) - \alpha_3||T_p(\mu,\rho,t) - T_p(\rho,\mu,t)|$$
$$\le |\alpha_1 + \alpha_4||T_p(\mu,\rho,t) - T_p(\rho,\mu,t)| \quad (3)$$

Here, $|\alpha_1 + \alpha_4| < 1$, above inequality hold.

$$\Rightarrow T_p(\mu,\rho,t) - T_p(\rho,\mu,t) = 0 \quad (4)$$

From Eqns. (1), (2) and (4), we have

$$T_p(\mu,\rho,t) = 0 \ \text{ and } \ T_p(\rho,\mu,t) = 0$$

$\Rightarrow \mu = \rho$
This completes the proof.

Theorem 3.2: Let complete parametric metric space is (A, T_p) and $T: A \to A$ be a continuous then it satisfying the below condition:

$$T_p(Ta, Tb, t) \le \alpha_1 T_p(a, b, t) + \alpha_2[T_p(a, Ta, t) + T_p(b, Tb, t)]\left[\frac{T_p(a, b, t) + T_p(b, Tb, t)}{T_p(a, Tb, t)}\right] + \alpha_3[T_p(a, Tb, t) + T_p(b, Ta, t)]\left[\frac{T_p(a,b,t)+T_p(b,Tb,t)+T_p(a,Tb,t)}{T_p(a,Tb,t)}\right] \quad (5)$$

where $\alpha_1, \alpha_2, \alpha_3 \ge 0$ with $\alpha_1 + 2\alpha_2 + 8\alpha_3 < 1$ for all $a, b \in A$ and $t > 0$ .Then T has unique fixed point.

Proof: Let m_0 be an initial point and a sequence $\{m_j\}$ such that $m_j = Tm_{j-1} = T^j m_0$. If there is a point $m_0 \in A$ such that $m_j = m_{j+1}$, then m_j is a fixed point. Therefore there is no need to precede further, Otherwise $m_j \ne m_{j+1}$. Using the inequality (1), we have

$$T_p(a,b,t) = T_p(Tm_j, Tm_{j+1}, t) \le \alpha_1 T_p(m_j, m_{j+1}, t) + \alpha_2[T_p(m_j, Tm_j, t) + T_p(m_{j+1}, Tm_{j+1}, t)]\left[\frac{T_p(m_j, m_{j+1}, t) + T_p(m_{j+1}, Tm_{j+1}, t)}{T_p(m_j, Tm_{j+1}, t)}\right] + \alpha_3[T_p(m_j, Tm_{j+1}, t) + T_p(m_{j+1}, Tm_j, t)]\left[\frac{T_p(m_j, m_{j+1}, t) + T_p(m_{j+1}, Tm_{j+1}, t) + T_p(m_j, Tm_{j+1}, t)}{T_p(m_j, Tm_{j+1}, t)}\right]$$

$$\le \alpha_1 T_p(m_j, m_{j+1}, t) + \alpha_2[T_p(m_j, m_{j+1}, t) + T_p(m_{j+1}, m_{j+2}, t)]\left[\frac{T_p(m_j, m_{j+1}, t) + T_p(m_{j+1}, m_{j+2}, t)}{T_p(m_j, m_{j+2}, t)}\right] + \alpha_3[T_p(m_j, m_{j+2}, t) + T_p(m_{j+1}, m_{j+1}, t)]\left[\frac{T_p(m_j, m_{j+1}, t) + T_p(m_{j+1}, m_{j+2}, t) + T_p(m_j, m_{j+2}, t)}{T_p(m_j, m_{j+2}, t)}\right]$$

$$\le \alpha_1 T_p(m_j, m_{j+1}, t) + \alpha_2[T_p(m_j, m_{j+1}, t) + (T_p(m_{j+1}, m_{j+2}, t)]\left[\frac{T_p(m_j, m_{j+1}, t) + T_p(m_{j+1}, m_{j+2}, t)}{T_p(m_j, m_{j+2}, t)}\right] + \alpha_3[T_p(m_j, m_{j+2}, t) + T_p(m_{j+1}, m_{j+2}, t) + T_p(m_j, m_{j+1}, t)][2\frac{T_p(m_j, m_{j+1}, t)}{T_p(m_j, m_{j+2}, t)}]$$

$$\le \alpha_1 T_p(m_j, m_{j+1}, t) + \alpha_2[T_p(m_j, m_{j+1}, t) + T_p(m_{j+1}, m_{j+2}, t)] + 4\alpha_3[T_p(m_j, m_{j+1}, t) + T_p(m_{j+1}, m_{j+2}, t)]$$

$$T_p(m_{j+1}, m_{j+2}, t) \le \frac{\alpha_1 + \alpha_2 + 4\alpha_3}{1 - 4\alpha_3 - \alpha_2} T_p(m_j, m_{j+1}, t)$$

Let $h = \frac{\alpha_1+\alpha_2+4\alpha_3}{1-4\alpha_3-\alpha_2}$ $as\ h < 1$ $\quad h = \alpha_1 + 2\alpha_2 + 8\alpha_3 < 1$

therefore, $T_p(m_{j+1}, m_{j+2}, t) \le hT_p(m_j, m_{j+1}, t)$,

Similarly $T_p(m_j, m_{j+1}, t) \le hT_p(m_{j-1}, m_j, t)$

$$T_p(m_{j+1}, m_{j+2}, t) \le h^2 T_p(m_{j-1}, m_j, t)$$

Using iteration up to j times,

$$T_p(m_j, m_{j+1}, t) \le h^j T_p(m_0, m_1, t)$$

where $0 \le h \le 1$ and t>0 $\Rightarrow h^j \to 0$ as $j \to \infty$.Using lemma 2.5 sequence $\{m_j\}$ is Cauchy sequence. So $\exists\ \mu \in m$ such that $m_j \to \mu$ as $j \to \infty$. Now, we will prove $\mu$is a fixed point of T.

Since $m_j \to \mu\ as\ j \to \infty$.By means of continuity , we have

$$\lim_{j\to\infty} Tm = T\mu$$
$$\lim_{j\to\infty} m_{j+1} = T\mu$$

Then $T\mu = \mu$, then μ is a fixed point of T.

For uniqueness, let us consider μ and ρ be the two fixed point of T for$\mu \ne \rho$, we have

$$T_p(\mu, \rho, t) \le \alpha_1 T_p(\mu, \rho, t) + \alpha_2[T_p(\mu, T\mu, t) + T_p(\rho, T\rho, t)]\left[\frac{T_p(\mu, \rho, t) + T_p(\rho, T\rho, t)}{T_p(\mu, T\rho, t)}\right] + \alpha_3[T_p(\mu, T\rho, t) + T_p(\mu, T\mu, t)][\frac{T_p(\mu, \rho, t) + T_p(\rho, T\rho, t) + T_p(\mu, T\rho, t)}{T_p(\mu, T\rho, t)}]$$

Here, μ and ρ are fixed point of T.

Therefore, by given condition, we have

$$T_p(\mu, \mu, t) = 0 \text{ and } T_p(\rho, \rho, t) = 0$$

So, above equation become

$$T_p(\mu, f_2{}^*, t) \le \alpha_1 T_p(\mu, f_2{}^*, t) + 2\alpha_3 T_p(\mu, f_2{}^*, t) + 2\alpha_3 T_p(f_2{}^*, \mu, t)$$

And $(\mu, \rho, t) \le (\alpha_1 + 2\alpha_3) T_p(\mu, \rho, t) + 2\alpha_3 T_p(\rho, \mu, t)$ (6)

Similarly,

$$T_p(\rho, \mu, t) \le (\alpha_1 + 2\alpha_3) T_p(\rho, \mu, t) + 2\alpha_3 T_p(\mu, \rho, t) \quad (7)$$

Subtract above two equations, we get

$$|T_p(\mu, \rho, t) - T_p(\rho, \mu, t)| \le |\alpha_1 + 2\alpha_3 - 2\alpha_3||T_p(\mu, \rho, t) - T_p(\rho, \mu, t)| \quad (8)$$
$$\le |\alpha_1||T_p(\mu, \rho, t) - T_p(\rho, \mu, t)|$$

Clearly, $|\alpha_1| < 1$. So, above inequality holds.

If $T_p(\mu, \rho, t) - T_p(\rho, \mu, t) = 0$ (9)

From (5),(6)and (8),we have

$$T_p(\mu, \rho, t) = 0\ and T_p(\rho, \mu, t) = 0$$

$\Rightarrow \mu = \rho$

This completes the proof. T has unique fixed point.

Example 3.3: Consider(A, T_p) be a complete parametric metric space and $T: R^+ \to R^+$ be a mapping ,since

$T_p(x^*, y^*, t) = t \mid x^* - y^* \mid$ such that

$x^*{}_\delta = 1 + \frac{1}{\delta}$ and $y^*{}_\delta = 1 + \frac{2}{\delta}$

Therefore, $T_p(x^*{}_\delta, y^*{}_\delta, t) = t \mid x^*{}_\delta - y^*{}_\delta \mid$

$$= t \mid 1 + \frac{1}{\delta} - 1 - \frac{2}{\delta} \mid$$
$$= t \mid -\frac{1}{\delta} \mid = t\frac{1}{\delta}$$

$$\log_{\delta\to\infty} T_p(x^*{}_\delta, y^*{}_\delta, t) = \log_{\delta\to\infty} t\frac{1}{\delta} = t\delta = 0$$
$$= \log_{\delta\to\infty} T_p(x^*{}_\delta, y^*{}_\delta, t) \to 0$$

As both, $x^*{}_\delta = 1 + \frac{1}{\delta}$ and $y^*{}_\delta = 1 + \frac{2}{\delta}$ tend to 1 as $\delta \to \infty$.Hence 1 is the fixed point.

Hence it satisfy all the condition of complete parametric metric space for $t > 0$. Note that for Theorem 3.1: $\alpha_1 = \frac{1}{4}, \alpha_2 = \frac{1}{3}, \alpha_3 = \frac{1}{9}, \alpha_4 = \frac{17}{12}, \alpha_5 = \frac{1}{18}$ and Theorem 3.2:$\alpha_1 = \frac{1}{8}, \alpha_2 = \frac{1}{16}, \alpha_3 = \frac{1}{32}$

Theorem 3.4: Let (A, T_p) be a complete parametric metric space and $t > 0$. Let $S, T: A \to A$ be a mapping then it satisfy the condition:

1. $T(A) \subseteq S(A)$
2. S,T is continuous and
3. $T_p(Sx, Ty) \le \alpha_1 T_p(x, y, t) + \alpha_2[T_p(x, Sx, t) + T_p(y, Ty, t)]\left[\frac{T_p(x,y,t)+T_p(y,Ty,t)}{T_p(x,Ty,t)}\right] + \alpha_3[T_p(x, Ty, t) + T_p(y, Sx, t)][\frac{\{T_p(x,y,t)+T_p(y,Ty,t)+T_p(x,Ty,t)\}^2}{T_p(x,Ty,t)^2}]$ (10)

Where $\alpha_1 + \alpha_2 + \alpha_3 \geq 0$ with $\alpha_1 + 2\alpha_2 + 12\alpha_3 < 1 \; \forall \; x, y \in A$ and $t > 0$. Then prove that S,T has a common unique fixed point.

Proof: Let $m_0 \in A$ be any arbitrary point and the sequence $\{m_j\}_{j\epsilon N}$,we have

$m_1 = S(m_0), m_2 = T(m_1) \ldots\ldots m_{2j+1} = Sm_{2j}, m_{2j} = T(m_{2j-1})$, we have

$$T_p(m_{2j+1}, m_{2j+2}, t) = T_p(Sm_{2j}, Tm_{2j+1}, t)$$
$$\leq \alpha_1 T_p(m_{2j}, m_{2j+1}, t)$$
$$+\alpha_2[T_p(m_{2j}, Sm_{2j}, t) + T_p(m_{2j+1}, Tm_{2j+1}, t)]$$
$$\left[\frac{T_p(m_{2j}, m_{2j+1}, t) + T_p(m_{2j+1}, Tm_{2j+1}, t)}{T_p(m_{2j}, Tm_{2j+1}, t)}\right]$$
$$+\alpha_3[T_p(m_{2j}, Tm_{2j+1}, t) + T_p(m_{2j+1}, Sm_{2j}, t)]$$
$$\left[\frac{\left\{\begin{matrix} T_p(m_{2j}, m_{2j+1}, t) + T_p(m_{2j+1}, Tm_{2j+1}, t) \\ +T_p(m_{2j}, Tm_{2j+1}, t) \end{matrix}\right\}^2}{\{T_p(m_{2j}, Tm_{2j+1}, t)\}^2}\right]$$
$$\leq \alpha_1 T_p(m_{2j}, m_{2j+1}, t)$$
$$+\alpha_2[T_p(m_{2j}, m_{2j+1}, t) + T_p(m_{2j+1}, m_{2j+2}, t)]$$
$$\left[\frac{T_p(m_{2j}, m_{2j+1}, t) + T_p(m_{2j+1}, m_{2j+2}, t)}{T_p(m_{2j}, m_{2j+2}, t)}\right]$$
$$+\alpha_3[T_p(m_{2j}, m_{2j+2}, t) + T_p(m_{2j+1}, m_{2j+1}, \square)]$$
$$\left[\frac{\left\{\begin{matrix} T_p(m_{2j}, m_{2j+1}, t) + T_p(m_{2j+1}, m_{2j+2}, t) \\ +T_p(m_{2j}, m_{2j+2}, t) \end{matrix}\right\}^2}{\{T_p(m_{2j}, m_{2j+2}, t)\}^2}\right]$$
$$\leq \alpha_1 T_p(m_{2j}, m_{2j+1}, t)$$
$$+\alpha_2[T_p(m_{2j}, m_{2j+1}, t) + T_p(m_{2j+1}, m_{2j+2}, t)]$$
$$+ \alpha_3[T_p(m_{2j}, m_{2j+2}, t) + T_p(m_{2j+1}, m_{2j+1}, t)][\{2\}^2]$$
$$\leq \alpha_1 T_p(m_{2j}, m_{2j+1}, t)$$
$$+\alpha_2[T_p(m_{2j}, m_{2j+1}, t) + T_p(m_{2j+1}, m_{2j+2}, t)]$$
$$+4\alpha_3\left[\begin{matrix} T_p(m_{2j}, m_{2j+1}, t) + T_p(m_{2j+1}, m_{2j+2}, t) \\ +T_p(m_{2j}, m_{2j+1}, t) + T_p(m_{2j+1}, m_{2j+2}, t) \end{matrix}\right]$$
$$\leq \alpha_1 T_p(m_{2j}, m_{2j+1}, t) + \alpha_2 T_p(m_{2j}, m_{2j+1}, t) + 12\alpha_3 T_p(m_{2j}, m_{2j+1}, t)$$
$$\leq \frac{\alpha_1 + \alpha_2 + 12\alpha_3}{1 - \alpha_2 - 4\alpha_3} T_p(m_{2j}, m_{2j+1}, t)$$
$$\Rightarrow T_p(m_{2j+1}, m_{2j+2}, t) \leq k(m_{2j}, m_{2j+1}, t)$$

Where, $k = \frac{\alpha_1+\alpha_2+12\alpha_3}{1-\alpha_2-4\alpha_3}$; $0 < k < 1$

Continue in this way, we have

$$T_p(m_{2j+1}, m_{2j+2}, t) \leq k^{2j} T_p(m_{2j}, m_{2j+1}, t) \quad ; 0 < k < 1$$

$k^{2j} \to 0 \; as j \to \infty$.Using lemma 2.5 sequence $\{m_j\}_{j\epsilon N}$ is Cauchy sequence. Thus, $\exists \; \mu \in A \; s.t \{m_j\}$ converges to μ. Further the sub-sequence $\{Sm_{2j}\} \to \mu$ and $\{Tm_{2j}\} \to \mu$ .Since $S, T: A \to A$ are continuous, we have

$$S\,\mu = \mu \; and \; T\,\mu = \mu$$

Then, μ is a fixed point of S and T.

$$\Rightarrow S\,\mu = \mu = T\,\mu$$

Now, for uniqueness μ and ρ be the two-fixed point of S and T, then we get

$$T_p(\mu, \rho, t) = T_p(S\,\mu, T\rho, t) \;\; \leq \alpha_1 T_p(\mu, \rho, t)$$
$$+\alpha_2[T_p(\mu, S\,\mu, t) + T_p(\rho, \rho, t)\left[\frac{T_p(\mu, \rho, t) + T_p(\rho, T\rho, t)}{T_p(\mu, T\rho, t)}\right]$$
$$+\alpha_3[T_p(\mu, T\rho, t) + T_p(\rho, S\,\mu, t)$$
$$[\frac{T_p(\mu, \rho, t) + T_p(\rho, T\rho, t) + T_{\square}(\mu, T\rho, t)\}^2}{[T_p(\mu, T\rho, t)]^2}$$
$$\leq \alpha_1 T_p(\mu, \rho, t) + \alpha_2[T_p(\mu, \mu, t)$$
$$+ T_p(\rho, \rho, t)\left[\frac{T_p(\mu, \rho, t) + T_p(\rho, \rho, t)}{T_p(\mu, \rho, t)}\right] + \alpha_3[T_p(\mu, \rho, t)$$
$$+ T_p(\rho, \mu, t)[\frac{T_p(\mu, \rho, t) + T_p(\rho, \rho, t) + T_p(\mu, \rho, t)\}^2}{[T_p(\mu, \rho, t)]^2}]$$

Hence $T_p(\mu, \mu, t) = 0 \;\; an \;\; T_p(\rho, \rho, t) = 0$

$$\leq \alpha_1 T_p(\mu, \rho, t) + \alpha_3[T_p(\mu, \rho, t)$$
$$+ T_p(\rho, \mu, t)[\frac{T_p(\mu, \rho, t) + T_p(\mu, \rho, t)\}^2}{[T_p(\mu, \rho, t)]^2}]$$
$$\leq \alpha_1 T_p(\mu, \rho, t) + 4\alpha_3[T_p(\mu, \rho, t) + \alpha_3 T_p(\rho, \mu, t)]$$
$$\leq (\alpha_1 + 4\alpha_3) T_p(\mu, \rho, t) + 4\alpha_3 T_p(\rho, \mu, t)$$
$$T_p(\mu, \rho, t) \leq (\alpha_1 + 4\alpha_3) T_p(\mu, \rho, t) + 4\alpha_3 T_p(\rho, \mu, t) \qquad (11)$$

$$T_p(\rho, \mu, t) \leq (\alpha_1 + 4\alpha_3) T_p(\rho, \mu, t) + 4\alpha_3 T_p(\mu, \rho, t) \qquad (12)$$

Subtract above two equations, we have

$$|T_p(\mu, \rho, t) - T_p(\rho, \mu, t)|$$
$$\leq |\alpha_1 + 4\alpha_3 - 4\alpha_3||T_p(\mu, \rho, t) - T_p(\rho, \mu, t)|$$
$$\leq |\alpha_1||T_p(\mu, \rho, t) - T_p(\rho, \mu, t) \qquad (13)$$

Clearly, $|\alpha_1| < 1$. So, above inequality holds.

$$\Rightarrow T_p(\mu, \rho, t) - T_p(\rho, \mu, t) = 0 \qquad (14)$$

From (11),(12)and (14), we have

$$T_p(\mu, \rho, t) = 0 \;\; and \;\; T_p(\rho, \mu, t) = 0$$

$\Rightarrow \mu = \rho$.

This completes the proof. T and S have unique common fixed point

IV. CONCLUSIONS

In this paper, we proved two common fixed point theorems in parametric metric space by using the various expansive contraction conditions. We found that this point is unique if the mapping is complete. Further we proved a fixed point theorem for double maps in parametric.

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