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Fixed Point Theorems on Complete Metric Space

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ABSTRACT: In the current paper, we prove fixed point theorems for self-mapping satisfying contraction condition on complete metric spaces. We used linear and rational expressions in our results which are the extensions of previous results proved by other authors.

Keywords: Fixed point, Contraction mapping, Cauchy sequence, Complete metric space.

I. INTRODUCTION

Fixed point theory is an energizing part of science. It is a blend of analysis, topology and geometry. In the course of the most recent 60 years or so the theorem of fixed points has been discovered as a significant device in the investigation of nonlinear phenomenon. It has various applications in practically all zones of scientific sciences. For example, integral equations, system of linear equations, proving the existence of solutions of ordinary and partial differential equations etc. The idea of fixed point assumes a key job in examination. Likewise, fixed point theorems are chiefly utilized in presence hypothesis of arbitrary differential conditions, numerical strategies like Newton-Rapshon method and Picard's Existence Theorem and in other related regions. In specific, fixed point techniques have been useful in such a miscellaneous field as biology, engineering and physics. After Banach Contraction Principle [12], the study of existence and uniqueness of fixed point and fixed point have been a main area of concentration. A fixed point is a point which does not change on application of mapping, differential equations system etc. .In fixed point theory, the most important result is called Banach Contraction Principle [12] (BCP) or Banach's fixed point theorem, which was given by Stefan Banach in 1922. This principle state that "if (N, d) is a complete metric space and $A: N \rightarrow N$ is a contraction mapping that is $d(Ar, As) \leq k\, d(r,\ s)$ where $k \in\ (0,1)$ for all $r, s \in\ N$ has a unique fixed point(every contraction mapping defined on a complete metric space has a fixed point.) ". Since its inceptions in 1922, this contraction principle has seen many extension and generalization in different space. It is the simplest and one of the most versatile results in fixed point theory. In 1994, Steve G Matthew [9] extended the metric space in Partial Metric Space (PMS). In PMS the distance of two respect points is not zero. Some of the various contributors in the study of stability of the fixed-point iterative schemes are Ostrowski [23], Harder and Hicks [24], Rhoades [26-34], Osilike [35], Berinde [36-37]. These authors used the method of the summability theory of infinite matrices to prove various stability results for certain definitions. Here, we have proved a fixed-point theorem in complete metric space with the help of Banach Contraction Theorem.

II. PRELIMINARIES

Before going to the main results, here are some definitions, lemmas, properties and examples in sequel, most of are taken from the work of [15, 17, 1, 3]. All the basic definitions will be useful for us to understand the work presented in the next section.

Definition 2.1: Let $X \neq \emptyset$ and a self-map be $T: X \rightarrow X$; a point $p \in X$ s.t. $Tp = p$ is known as a fixed point.

Definition 2.2: The metric space X is known as complete if each cauchy sequence →X i.e. if $d(x_n, x_m) \rightarrow 0$ as m, n approaches to infinity, then there is some $z \in X$ with $d(x_n, z) \rightarrow 0$

Definition 2.3: The sequence $\{x_n\}$ in (X, p) metric space is defined as convergent to a point $x \in X$ denoted by $\lim_{n\to\infty} x_n = x$ if $\lim_{n\to\infty} p(x_n, x) = 0.$

Definition 2.4: Suppose $X \neq \emptyset$ and there is a mapping $d: X \rightarrow X$ and it belongs to the set of real numbers. Then, it is said as metric (or distance function) iff fulfils the accompanying aphorisms

(M-l) $d(x, y) \geq 0$ for all $x, y \in X$
(M-2)$d(x, y) = 0$ iff $x = y$
(M-3)$d(x, y) = d(y, x) \forall\ x,y \in X$
(M-4)$d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$

If d is metric for X, then the ordered pair (X, d) is known as metric space and $d(x,y)$ is defined as distance between x and y.

Definition 2.5: Let $(X, d) \neq \emptyset$ and complete metric space defined by contraction mappings $T: X \rightarrow X$. Then, T acknowledges unique fixed point σ^* in X such that $T(\sigma^*) = \sigma^*$.

Lemma 2.6: Let us assume a sequence $\{x_n\}$ in (X, d) metric space such that

$$d(x_n, x_{n+1}) \leq l d(x_{n-1}, x_n)$$

Where, $l \in [0,1)$ and $n = 1,2,3 \ldots..$
Then $\{x_n\}$ be a Cauchy sequence in (X, d)

Verification: Let $n > m \geq 1$, it follows that

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \dots \dots d(x_{m-1}, x_m) \leq (l^m + l^{m+1} \dots \dots . l^{m-1}) d(x_0, x_1)$$

Since $l < 1$. Assume that $d(x_0, x_1) > 0$.
By taking $\lim_{m,n \to +\infty}$in above inequality, we get

$$\lim_{m,n \to +\infty} d\,(x_m, x_n) = 0$$

Therefore, $\{x_n\}$ be Cauchy sequence in X.
Also, if $d(x_0, x_1) = 0$ then $d(x_m, x_n) = 0 \;\forall\, m > n$
Hence $\{x_n\}$ be a Cauchy sequence in X.
Definition 2.8 [4] The self-mapping R in (X, d) metric space is called as Lipschitzian if $\forall\, x, y \in X$ and $\sigma \geq 0$ as such $d(f(x), f(y)) \leq \sigma\, d(x, y)$.
(a) R is said to be contraction on σ if $\sigma \in [0,1)$.
(b) R is non − expansive if $\sigma = 1$, i.e. $d(f(x), f(y)) < d(x, y)$.
(c) R is contractive if $\sigma < 1$.

III. MAIN RESULTS

Theorem 3.1 Let (X, d) be a CMS. A mapping $T: X \to X$ is such that

$$d(Tx, Ty) \leq \alpha_1 d(x,y) + \alpha_2[d(x,Tx) + d(y,Ty)] + \alpha_3[d(x,Ty) + d(y,Tx)] + \alpha_4\left[\frac{d(x,y)d(x,Ty)}{d(x,y) + d(y,Ty)}\right] + \alpha_5\left[\frac{d(x,Ty)d(y,Ty)}{d(x,y)+d(y,Ty)}\right] \quad (1)$$

$\forall\, x, y \in X; x \neq y$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in [0,1)$ and such that$\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 < 1$ and $\alpha_1 + 2\alpha_3 < 1$.
Then, T contains a fixed point which is also unique in X.
Proof Let a sequence be $\{x_n\}$ in X defined as for $x_0 \in X, Tx_n = x_{n+1}$ for all n = 0,1,2 .... Using (1), we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \alpha_1[d(x_{n-1}, x_n)] + \alpha_2[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] + \alpha_3[d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})] + \alpha_4\left[\frac{d(x_{n-1}, x_n)d(x_{n-1}, Tx_n)}{d(x_{n-1}, x_n) + d(x_n, Tx_n)}\right] + \alpha_5\left[\frac{d(x_{n-1}, Tx_n)d(x_n, Tx_n)}{d(x_{n-1}, x_n) + d(x_n, Tx_n)}\right]$$

$$\leq \alpha_1[d(x_{n-1}, x_n)] + \alpha_2[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + \alpha_3[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] + \alpha_4\left[\frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}\right] + \alpha_5\left[\frac{d(x_{n-1}, x_{n+1})d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}\right]$$

$$\leq \alpha_1[d(x_{n-1}, x_n)] + \alpha_2[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + \alpha_3[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + \alpha_4\left[\frac{d(x_{n-1}, x_n)[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]}{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}\right] + \alpha_5\left[\frac{[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}\right]$$

$$\leq \alpha_1[d(x_{n-1}, x_n)] + \alpha_2[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + \alpha_3[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + \alpha_4[d(x_{n-1}, x_n)] + \alpha_5[d(x_n, x_{n+1})]$$

$$(1 - \alpha_2 - \alpha_3 - \alpha_5)[d(x_n, x_{n+1})] \leq (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)[d(x_{n-1}, x_n)]$$

$$\Rightarrow d(x_n, x_{n+1}) \leq \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{1 - \alpha_2 - \alpha_3 - \alpha_5} d(x_{n-1}, x_n)$$

$$\Rightarrow d(x_n, x_{n+1}) \leq k\, d(x_{n-1}, x_n)$$

$k = \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{1 - \alpha_2 - \alpha_3 - \alpha_5} < 1$,Since $\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 < 1$.
Therefore, we have
$d(x_n, x_{n+1}) \leq k\, d(x_{n-1}, x_n) \leq k^2 d(x_{n-2}, x_{n-1}) \leq \cdots$
Continuing this process up to n iterates. We have

$$d(x_n, x_{n+1}) \leq k^n . d(x_0, x_1)$$

As $0 \leq k \leq 1$, so for $n \to \infty, k^n \to 0$
therefore $d(x_n, x_{n+1}) \to 0$.

Thus, $\{x_n\}$ is Cauchy sequence in CMS X.

So, there is a point $x^* \in X$ such that $x_n \to x^*$.

As T is continuous,

$$T(x^*) = \lim T(x_n) = \lim x_{n+1} = x^*$$

Hence, T has a fixed point.
If possible, consider y^* be another fixed point.
Using (1), we have
$d(x^*, y^*) = d(Tx^*, Ty^*)$

$$d(x^*, y^*) \leq \alpha_1[d(x^*, y^*)] + \alpha_2[d(x^*, Tx^*) + d(y^*, Ty^*)] + \alpha_3[d(x^*, Ty^*) + d(y^*, Tx^*)] + \alpha_4\left[\frac{d(x^*, y^*)d(x^*, Tx^*)}{d(x^*, y^*) + d(y^*, Ty^*)}\right] + \alpha_5\left[\frac{d(x^*, Ty^*)d(y^*, Ty^*)}{d(x^*, y^*) + d(y^*, Ty^*)}\right]$$

As, x^* and y^* are fixed points.
Thus,

$$d(x^*, y^*) \leq \alpha_1[d(x^*, y^*)] + \alpha_2[d(x^*, x^*) + d(y^*, y^*)] + \alpha_3[d(x^*, y^*) + d(y^*, x^*)] + \alpha_4\left[\frac{d(x^*, y^*)d(x^*, x^*)}{d(x^*, y^*) + d(y^*, y^*)}\right] + \alpha_5\left[\frac{d(x^*, x^*)d(y^*, y^*)}{d(x^*, y^*) + d(y^*, y^*)}\right]$$

$$\leq \alpha_1 d(x^*, y^*) + \alpha_2[0] + \alpha_3[2. d(x^*, y^*)] + \alpha_4[0] + \alpha_5[0]$$

$$d(x^*, y^*) \leq (\alpha_1 + 2.\alpha_3)d(x^*, y^*)$$

$$[1 - \alpha_1 - 2.\alpha_3]d(x^*, y^*) \leq 0$$

is a contradiction.

$$\Rightarrow d(x^*, y^*) = 0$$

$\Rightarrow x^*$ and y^* are not different points but are same
This completes the proof.

Theorem 3.2 Let a CMS be (X, d). Suppose $T: X \to X$ is a self-mapping which satisfies

$$d(T(x), T(y)) \leq \alpha_1 d(x, y) + \alpha_2[d(x, Tx) + d(y, Ty)]\left[\frac{d(x,y) + d(y,Ty)}{d(x,Ty)}\right] + \alpha_3[d(x, Ty) + d(y, Tx)]\left[\frac{d(x,y)+d(y,Ty)+d(x,Ty)}{d(x,Ty)}\right] \quad (1)$$

for all $x, y \in X$; $i \neq j$ and $\alpha_1, \alpha_2, \alpha_3 \in [0,1)$ such that $\alpha_1 + 2\alpha_2 + 4\alpha_3 < 1$ and $\alpha_1 + 2\alpha_3 < 0$.Then, T contains a fixed point which is also unique in X
Proof Let a sequence be $\{x_n\}$ in X, define $x_0 \in X$, s.t.$Tx_n = x_{n+1}$ for all n = 0, 1, 2…. Using (1), we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)]\left[\frac{d(x_{n-1}, x_n) + d(x_n, Tx_n)}{d(x_{n-1}, Tx_n)}\right] + \alpha_3[d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})]\left[\frac{d(x_{n-1}, x_n) + d(x_n, Tx_n) + d(x_{n-1}, Tx_n)}{d(x_{n-1}, Tx_n)}\right]$$

$$\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\left[\frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_{n+1})}\right] + \alpha_3[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)]\left[\frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_{n-1}, x_{n+1})}{d(x_{n-1}, x_{n+1})}\right]$$

$$\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + \alpha_3[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_{n-1}, x_{n+1})]$$

$$d(x_n, x_{n+1}) \leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + \alpha_3[2d(x_{n-1}, x_n) + 2d(x_n, x_{n+1})]$$

$$d(x_n,x_{n+1}) \le \frac{[\alpha_1+\alpha_2+2\alpha_3]}{[1-\alpha_2-2\alpha_3]} d(x_{n-1},x_n)$$
$$\Rightarrow d(x_n,x_{n+1}) \le r\, d(x_{n-1},x_n)$$
Where $r = \frac{\alpha_1+\alpha_2+2\alpha_3}{1-\alpha_2-2\alpha_3} < 1$; Since, $\alpha_1+2\alpha_2+4\alpha_3 < 1$
Repeating iteration, we have
$d(x_n,x_{n+1}) \le r\, d(x_{n-1},x_n) \ldots \le r^n d(x_0,x_1)$
As $0 \le r \le 1$, so for $n \to \infty, r^n \to 0$
therefore $d(x_n,x_{n+1}) \to 0$.
$\{x_n\}$ be a Cauchy sequence in CMS X.
Thus, there is a point $e \in X$ as such $x_n \to e$.
As, T is continuous,
$$T(e) = \lim T(x_n) = \lim x_{n+1} = e$$
Hence, there is a fixed point in T.
If possible, let us say f be any other fixed point. Using (1), we get

$$d(e,f) \le \alpha_1 d(e,f)$$
$$+\alpha_2[d(e,Tf)+d(f,Tf)]\left[\frac{d(e,f)+d(f,Tf)}{d(e,Tf)}\right]$$
$$+\alpha_3[d(e,Tf)+d(f,Te)]\left[\frac{d(e,f)+d(f,Tf)+d(e,Tf)}{d(e,Tf)}\right]$$
$$\le \alpha_1 d(e,f)+\alpha_2[d(e,f)+d(f,f)]\left[\frac{d(e,f)+d(f,f)}{d(e,f)}\right]$$
$$+\alpha_3[d(e,f)+d(f,e)]\left[\frac{d(e,f)+d(f,f)+d(e,f)}{d(e,f)}\right]$$
$$\le \alpha_1 d(e,f)+\alpha_2(0)+2\alpha_3 d(e,f)$$
is a contradiction.
$$\Rightarrow d(e,f) = 0$$
Therefore, e = f
This completes the proof.

Theorem 3.3 Let us take a CMS (X,d). Let $S,T: X \to X$ then, it satisfies the below condition:
1. $T(X) \subseteq S(X)$
2. S, T is continuous and
3. $d(Sx,Ty) \le \alpha_1 d(x,y)+\alpha_2[d(x,Sx)+d(y,Ty)]\left[\frac{d(x,y)+d(y,Ty)}{d(x,Ty)}\right]+\alpha_3[d(x,Ty)+d(y,Sx)]\left[\frac{\{d(x,y)+d(y,Ty)+d(x,Ty)\}^2}{[d(x,Ty)]^2}\right]$ (1)

Where $\forall x,y \in X$.Then show that S, T contains unique fixed point.

Proof Let there be a arbitrary point $x_0 \in X$, defining the sequence $\{x_j\}_{j\epsilon N}$
$$x_1 = S(x_0)$$
$$x_2 = T(x_1)$$
$$\ldots\ldots$$
$$x_{2j+1} = Sx_{2j}$$
$$x_{2j} = T(x_{2j-1})$$
we have,
$$d(x_{2j+1},x_{2j+2}) = d(Sx_{2j},Tx_{2j+1})$$
$$\le \alpha_1 d(x_{2j},x_{2j+1})$$
$$+\alpha_2[d(x_{2j},Sx_{2j})+d(x_{2j+1},Tx_{2j+1})]\left[\frac{d(x_{2j},x_{2j+1})+d(x_{2j+1},Tx_{2j+1})}{d(x_{2j},Tx_{2j+1})}\right]$$
$$+\alpha_3[d(x_{2j},Tx_{2j+1})+d(x_{2j+1},Sx_{2j})]\left[\frac{\{d(x_{2j},x_{2j+1})+d(x_{2j+1},Tx_{2j+1})+d(x_{2j},Tx_{2j+1})\}^2}{\{d(x_{2j},Tx_{2j+1})\}^2}\right]$$

$$\le \alpha_1 d(x_{2j},x_{2j+1})+\alpha_2[d(x_{2j},x_{2j+1})+d(x_{2j+1},x_{2j+2})]\left[\frac{d(x_{2j},x_{2j+1})+d(x_{2j+1},x_{2j+2})}{d(x_{2j},x_{2j+2})}\right]$$
$$+\alpha_3[d(x_{2j},x_{2j+2})+d(x_{2j+1},x_{2j+1})]\left[\frac{\{d(x_{2j},x_{2j+1})+d(x_{2j+1},x_{2j+2})+d(x_{2j},x_{2j+2})\}^2}{\{d(x_{2j},x_{2j+2})\}^2}\right]$$

$$\le \alpha_1 d(x_{2j},x_{2j+1})+\alpha_2[d(x_{2j},x_{2j+1})+d(x_{2j+1},x_{2j+2})]+4.\alpha_3[d(x_{2j},x_{2j+2})]$$
$$\le \alpha_1 d(x_{2j},x_{2j+1})+\alpha_2\left[\begin{matrix} d(x_{2j},x_{2j+1}) \\ +d(x_{2j+1},x_{2j+2})\end{matrix}\right]+4.\alpha_3\left[\begin{matrix} d(x_{2j},x_{2j+1}) \\ +d(x_{2j+1},x_{2j+2})\end{matrix}\right]$$
$$\le \alpha_1[d(x_{2j},x_{2j+1})]+\alpha_2[d(x_{2j},x_{2j+1})+d(x_{2j+1},x_{2j+2})]+4\alpha_3[d(x_{2j},x_{2j+1})+d(x_{2j+1},x_{2j+2})]$$
$$d(x_{2j+1},x_{2j+2}) \le \frac{\alpha_1+\alpha_2+4\alpha_3}{1-\alpha_2-4\alpha_3} d(x_{2j},x_{2j+1})$$
$$d(x_{2j+1},x_{2j+2}) \le k(x_{2j},x_{2j+1})$$

Where, $k = \frac{\alpha_1+\alpha_2+4\alpha_3}{1-\alpha_2-4\alpha_3}$; $0 < k < 1$
Continuing in this way, we have
$$d(x_{2j+1},x_{2j+2}) \le k^{2j} d(x_{2j},x_{2j+1}) \quad ; 0 < k < 1$$
$k^{2j} \to 0$ as $j \to \infty$.
By lemma 2.7
$\{x_j\}_{j\epsilon N}$ is CMS in a Cauchy sequence.
Thus, $\exists\, l_1^* \in X$ s.t $\{x_j\}$, converges to l_1^*.
Further the subsequence
$$\{Sx_{2j}\} \to l_1^* \text{ and } \{Tx_{2j}\} \to l_1^*$$
Since $S,T: X \to X$ are continuous, we have
$$Sl_1^* = l_1^* \text{ and } Tl_1^* = l_1^*$$
Then, l_1^* is a fixed point of S and T.
$$\Rightarrow Sl_1^* = l_1^* = Tl_1^*$$
Let l_1^* and l_2^* be the fixed point of S and T, then we have
$$d(l_1^*,l_2^*) = d(Sl_1^*,Tl_2^*)$$
$$\le \alpha_1 d(l_1^*,l_2^*)+\alpha_2[d(l_1^*,l_1^*)+d(l_2^*,l_2^*)]\left[\frac{d(l_1^*,l_2^*)+d(l_2^*,l_2^*)}{d(l_1^*,l_2^*)}\right]$$

$$\le \alpha_1 d(l_1^*,l_2^*)$$
$$+\alpha_3[d(l_1^*,l_2^*)+d(l_2^*,l_1^*)]\left[\frac{\{d(l_1^*,l_2^*)+d(l_1^*,l_2^*)\}^2}{[d(l_1^*,l_2^*)]^2}\right]$$
$$\le \alpha_1 d(l_1^*,l_2^*)+4\alpha_3[d(l_1^*,l_2^*)]+4\alpha_3[d(l_2^*,l_1^*)]$$
$$\le (\alpha_1+4\alpha_3)d(l_1^*,l_2^*)+4\alpha_3 d(l_2^*,l_1^*)$$
$$d(l_1^*,l_2^*) \le (\alpha_1+4\alpha_3)d(l_1^*,l_2^*)+4\alpha_3 d(l_2^*,l_1^*) \quad (2)$$
Similarly,
$$d(l_2^*,l_1^*) \le (\alpha_1+4\alpha_3)d(l_2^*,l_1^*)+4\alpha_3 d(l_1^*,l_2^*) \quad (3.32)$$
Subtract above two equations, We have,
$$|d(l_1^*,l_2^*)-d(l_2^*,l_1^*)| \le |\alpha_1||d(l_1^*,l_2^*)-d(l_2^*,l_1^*)|$$
$$\le |\alpha_1||d(l_1^*,l_2^*)-d(l_2^*,l_1^*)| \quad (3)$$
Clearly, $|\alpha_1| < 1$
So, above inequality holds.
$$\Rightarrow d(l_1^*,l_2^*)-d(l_2^*,l_1^*) = 0 \quad (4)$$
From (2), (3) and (4), we have
$$d(l_1^*,l_2^*) = 0 \text{ and } d(l_2^*,l_1^*) = 0$$
$$\Rightarrow l_1^* = l_2^*$$
This completes the proof.

IV CONCLUSION

Here, some common theorems of fixed-points are proving on the basis of complete metric space by using rational type expression. We get the unique fixed point for single as well as pair of mapping in complete metric space. These outcomes can be stretched out to in excess of two mapping with extra conditions. Also, we can do research related to another metric spaces on this space.

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