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Statistical Λ −Convergence of Order α in Probabilistic Normed Spaces

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ABSTRACT: In this paper, we discuss the statistical Λ–convergence and statistical Λ–Cauchy in probabilistic normed space and extend this concept further for order α and present some basic results of statistical Λ–convergence for order α in probabilistic normed space by giving some illustrative examples.

Keywords: Statistical convergence, Statistical Λ-convergence, t-norm, Probabilistic Normed Space (PN −space)

I. INTRODUCTION

The term statistical convergence in general directly connected with an extension of the usual convergence concept for the sequences. This concept has been studied by various authors parallel to the theory of ordinary convergence. This important theory of convergence for the sequences was initiated by Fast [7] and Steinhaus [20] simultaneously. The advancement in this convergence began after the work of Schoenberg [17] and Šalát [16]. Later on numerous researchers worked on statistical convergence for different types of sequences and in different spaces like intuitionistic fuzzy normed space [11], random normed space [13], probabilistic normed space [9] etc.. The concept of statistical convergence was further explored using the summability theory by Fridy [8], Conner [6], Meenakshi *et al.*, [2], Šalát [16] etc. Çolak [3-5] extended the work on statistical convergence as well as $\lambda-$ statistical convergence for order α where the model of $\lambda-$ statistical convergence was presented by Mursaleen [12]. The basic structure of a statistical metric space was proposed by Menger [10], that is also known as probabilistic metric space which was later on explored by Schweizer and Sklar [18,19]. The term probabilistic metric space is applicable in those settings where exact distance between two points is unpredictable. Probabilistic normed space is considered as a very important class in the probabilistic metric space. Karakus [9] also demonstrated statistical convergence basic concepts in PN–spaces. Recently, Srivastava and Mursaleen [15] also associated their work on statistical $\Lambda-$ convergence in PN–spaces. Later, Meenakshi *et al.*, [1] also gave overview of Λ– statistical convergence fororder α in the setup of a random 2-normed space. Our present work generalize the statistical Λ–convergence in probabilistic normed space for order α.

II. PRELIMINARIES

First we review some basic terms required for our work which is based on natural density.

Definition 2.1 The term natural density of any set $\mathcal{M} \subseteq \mathrm{N}$ (set of natural numbers) is given as

$$\delta(\mathcal{M}) = \lim_{n\to\infty} \frac{1}{n} |\{a \le n : a \in \mathcal{M}\}| ,$$

where | . | represents the order of the enclosed set.

Definition 2.2 [7] A sequence $x = (x_m)$ is statistically convergent to ξ if for each $\varepsilon > 0$,

$$\mathcal{M}(\varepsilon) = \{m \le n : |x_m - \xi| > \varepsilon\}$$

has natural density zero. We can write it as St $\lim x_m = \xi$.

Definition 2.3 A function $\Phi : \mathcal{R} \to \mathcal{R}_+^0$ (set of positive real numbers) is known as a distribution function if it is a non-decreasing and left continuous when,

$\inf \Phi(c) = 0$; $c \in \mathcal{R}$,
$\sup \Phi(c) = 1$; $c \in \mathcal{R}$.
and $\mathcal{F}_+ = \{\Phi :\ \Phi \text{ is a distribution function with } \Phi(0) = 0\}$.

If $k \in \mathcal{R}_+^0$, then $\Phi'_k \in \mathcal{F}_+$, where $\Phi'_k(c) = \begin{cases} 0 & (c \le k) \\ 1 & (c > k) \end{cases}$, for all $\Phi \in \mathcal{F}_+$.

Definition 2.4 A t – norm (triangular form) is the map $*$: $[0,1] \times [0,1] \to [0,1]$ which is continuous, commutative, associative and non-decreasing.

Definition 2.5 [19] Let $\mathfrak{J} : \mathbb{X} \to \mathcal{F}_+$ be a map such that $\mathfrak{J}_x = \mathfrak{J}(x)$ and $\mathfrak{J}_x(p)$ is value of $\mathfrak{J}_x$ at $p \in \mathcal{R}_+^0$ (where $\mathbb{X}$ is a linear space and $\mathcal{F}_+$ is a set of distribution functions). Then $\mathfrak{J}$ is known as probabilistic norm and triplet $(\mathbb{X}; \mathfrak{J}, *)$ with $*$ as a t-norm is known as probabilistic normed space or PN–space if it satisfies below four properties:

(i) $\mathfrak{J}_x(0) = 0$,
(ii) $\mathfrak{J}_x(p) = 1$, $\forall p > 0$ iff $x = 0$,
(iii) $\mathfrak{J}_{x\mu}(p) = \mathfrak{J}_x\left({p}/{|\mu|}\right)$ where $\mu \neq 0 \in \mathcal{R}_+^0$,
(iv) $\mathfrak{J}_{x+y}(p+q) = \mathfrak{J}_x(p) * \mathfrak{J}_y(q)$, $\forall x, y \in \mathbb{X}$ and p, $q \in \mathcal{R}_+^0$.

Example 2.1 If $(\mathbb{X}, \|.\|)$ is a normed space, $\beta \in \mathcal{F}_+$ such that β (0) = 0 and $\beta \neq \Phi$ where

$\Phi(p) = \begin{cases} 1 & ,p > 0 \\ 0 & ,p \le 0 \end{cases}$, $x \in \mathbb{X}$ and $p \in \mathcal{R}$ with

$$\mathfrak{J}_x(p) = \begin{cases} \Phi(p) & , \ p = 0 \\ \beta\left(\frac{p}{\|x\|}\right) & , p \neq 0. \end{cases}$$

Then triplet $(\mathbb{X}; \mathfrak{J}, *)$ is known as a PN–space.

First we mention Λ – convergent sequence as given by Mursaleen [14]. Suppose $\lambda = (\lambda_i)_{i=0}^{\infty}$ is a real sequence with $\lim_{i\to\infty} \lambda_i \to \infty$ and $0 < \lambda_0 < \lambda_1 < \cdots < \lambda_i < \cdots$. In such case, a sequence $x = (x_m)_{m=0}^{\infty}$ is Λ-convergent to some number ξ when $\Lambda_m(x) \to \xi$ as $m \to \infty$, and

$$\Lambda_m(x) = \frac{1}{\lambda_m} \sum_{i=0}^{m} (\lambda_i - \lambda_{i-1})\, x_i, \ (m \in \mathrm{N}).$$

Next, we discuss the idea of Λ–convergence along with statistical Λ–convergence in PN–space.

Definition 2.6 [15] Let $(\mathbb{X};\Im,*)$ be a PN–space. A sequence $x=(x_m)_{m=0}^{\infty}$ is called convergent to some $\xi$with respect to $\Im$ in $(\mathbb{X};\Im,*)$ if for each $\varepsilon>0$ and $\Phi\in(0,1)$there exists a number $m_0\in N$ with $\Im_{x_m-\xi}(\varepsilon)>1-\Phi$ whenever $m\ge m_0$ and it can be written as,

$$\Im\lim_{m\to\infty}x_m=\xi.$$

Remark 2.1 Consider $(\mathbb{X},\|.\|)$ is a real normed space with $\Im_x(p)=\frac{p}{p+\|x\|}$, $p\ge 0$ and $x\in\mathbb{X}$ (t – norm ($*$) defined by $\|.\|$). Clearly, we observe that $x_m\xrightarrow{\|.\|}x$ iff $x_m\xrightarrow{\Im}x$.

Definition 2.7 [9] Let $(\mathbb{X};\Im,*)$ be a PN–space. A sequence $x=(x_m)_{m=0}^{\infty}$ is called statistically convergent to some $\xi$with respect to $\Im$ in $(\mathbb{X};\Im,*)$ if for each $\varepsilon>0$ and $\Phi\in(0,1)$, we have

$$\delta(\{m\in N:\Im(x_m-\xi)(\varepsilon)\le 1-\Phi\})=0.$$

Definition 2.8 [15] Let $(\mathbb{X};\Im,*)$ be a PN–space. A sequence $x=(x_m)_{m=0}^{\infty}$ is called Λ-convergent to some $\xi$with respect to $\Im$ in $(\mathbb{X};\Im,*)$ if for each $\varepsilon>0$ and $\Phi\in(0,1)$ there exists a number $m_0\in N$ with $\Im_{\Lambda x_m-\xi}(\varepsilon)>1-\Phi$ whenever $m\ge m_0$ and it can be written as,

$$\Im\lim_{m\to\infty}\Lambda x_m=\xi.$$

Definition 2.9 [15] Let $(\mathbb{X};\Im,*)$ be a PN-space. A sequence $x=(x_m)_{m=0}^{\infty}$ is called statistically Λ-convergent with respect to $\Im$ in $(\mathbb{X};\Im,*)$ if for each $\varepsilon>0$ and $\Phi\in(0,1)$, we have

$$\delta(\{m\in N:\Im(\Lambda x_m-\xi)(\varepsilon)\le 1-\Phi\})=0.$$

Symbolically, St. $\Im_\Lambda-\lim x_m=\xi$.

Definition 2.10 [9] Let $(\mathbb{X};\Im,*)$ be a PN–space. A sequence $x=(x_m)_{m=0}^{\infty}$ is called statistically Cauchy with respect to $\Im$ in $(\mathbb{X};\Im,*)$ if for each $\varepsilon>0$ there exists a number $m_0=m_0(\varepsilon)$ such that

$$\delta\left(\{m\in N:\Im\left(x_m-x_{m_0}\right)(\varepsilon)\le 1-\Phi\}\right)=0.$$

Definition 2.11 [15] Let $(\mathbb{X};\Im,*)$ be a PN–space. A sequence $x=(x_m)_{m=0}^{\infty}$ is called statistically Λ–Cauchy with respect to $\Im$ in $(\mathbb{X};\Im,*)$ if for each $\varepsilon>0$ there exists a number $m_0=m_0(\varepsilon)$ such that

$$\delta\left(\{m\in N:\Im\left(\Lambda x_m-\Lambda x_{m_0}\right)(\varepsilon)\le 1-\Phi\}\right)=0.$$

Definition 2.12 [3] Consider $\lambda=(\lambda_m)_{m=0}^{\infty}$ is a non-decreasing real sequence of positive numbers and $\alpha\in(0,1]$. A sequence $x=(x_m)_{m=0}^{\infty}$ is called statistically λ–convergent of order α to some ξ whenever,

$$\lim_{n\to\infty}\frac{1}{\lambda^\alpha}|\{m\in I_n:|x_m-\xi|\ge\varepsilon\}|=0,$$

where $I_n=[n-\lambda_n+1,n]$.

III. MAIN RESULTS

With the help of the definitions given in the previous segment, we systematically analyse the idea of statistical Λ–convergence as well as statistical Λ–Cauchy of order α in PN–space as follows:

Definition 3.1 Let $(\mathbb{X};\Im,*)$ be a PN–space and $\alpha\in(0,1]$. A sequence $x=(x_m)_{m=0}^{\infty}$ is called statistically Λ–convergent of order α or Λ^α – convergent to some $\xi$with respect to $\Im$ in $(\mathbb{X};\Im,*)$ if for each $\varepsilon>0$ and $\Phi\in(0,1)$, we have

$$\delta_{\Lambda^\alpha}(\{m\in N:\Im(\Lambda x_m-\xi)(\varepsilon)\le 1-\Phi\})=0,$$

i.e, $\lim_{n\to\infty}\frac{1}{n^\alpha}(\{m\in N:\Im(\Lambda x_m-\xi)(\varepsilon)\le 1-\Phi\})=0,$

or $\delta_{\Lambda^\alpha}(\{m\in N:\Im(\Lambda x_m-\xi)(\varepsilon)>(1-\Phi)\})=1,$

i.e, $\lim_{n\to\infty}\frac{1}{n^\alpha}(\{m\in N:\Im(\Lambda x_m-\xi)(\varepsilon)>1-\Phi\})=1.$

where ξ is called the statistical $\Im_\Lambda^\alpha$ limit .

Definition 3.2 Let $(\mathbb{X};\Im,*)$ be a PN–space and $\alpha\in(0,1]$. A sequence $x=(x_m)_{m=0}^{\infty}$ is called statistically Λ –Cauchy of order α or statistical $\Im_\Lambda^\alpha$ Cauchy with respect to $\Im$ in $(\mathbb{X};\Im,*)$ if for each $\varepsilon>0$ there exists a number $m_0=m_0(\varepsilon)$ such that

$$\delta_{\Lambda^\alpha}\left(\{m\in N:\Im\left(\Lambda x_m-\Lambda x_{m_0}\right)(\varepsilon)\le 1-\Phi\}\right)=0.$$

Theorem 3.1 Let $(\mathbb{X};\Im,*)$ be a PN–space and $\alpha\in(0,1]$. If a sequence $x=(x_m)_{m=0}^{\infty}$ is statistically Λ^α – convergent in PN space, then it has a unique limit.

Proof : Let us assume that, $\mathrm{St}\Im_\Lambda^\alpha-\lim x_m=\xi_1$ and $\mathrm{St}\Im_\Lambda^\alpha-\lim x_m=\xi_2$.

For given >0 , let $\Phi\in(0,1)$ and $\theta\in(0,1)$ such that $(1-\Phi)*(1-\Phi)>1-\theta$.

Since, $\delta_{\Lambda^\alpha}(\{m\in N:\Im(\Lambda x_m-\xi_1)(\varepsilon)\le 1-\Phi\})=0$ and $\delta_{\Lambda^\alpha}(\{m\in N:\Im(\Lambda x_m-\xi_2)(\varepsilon)\le 1-\Phi\})=0$.

i.e., $\delta_{\Lambda^\alpha}(M_1(\Phi,\varepsilon))=0$ and $\delta_{\Lambda^\alpha}(M_2(\Phi,\varepsilon))=0$.

where $M_1(\Phi,\varepsilon)=\{m\in N:\Im(\Lambda x_m-\xi_1)(\varepsilon)\le 1-\Phi\}$, $M_2(\Phi,\varepsilon)=\{m\in N:\Im(\Lambda x_m-\xi_2)(\varepsilon)\le 1-\Phi\}$.

Now, let $M(\Phi,\varepsilon)=M_1(\Phi,\varepsilon)\cup M_2(\Phi,\varepsilon)$.

Clearly, $\delta_{\Lambda^\alpha}(M(\Phi,\varepsilon))=0$.

This implies, $\delta_{\Lambda^\alpha}(M^c(\Phi,\varepsilon))=1$

i.e. $\delta_{\Lambda^\alpha}(\{m\in N:\Im(\Lambda x_m-\xi)(\varepsilon)>1-\Phi\})=1$.

For $m\in N-M(\Phi,\varepsilon)$, we get

$$\Im_{\xi_1-\xi_2}(\varepsilon)\ge\Im(\Lambda^\alpha x_m-\xi_1)\left(\frac{\varepsilon}{2}\right)*\Im(\Lambda^\alpha x_m-\xi_2)\left(\frac{\varepsilon}{2}\right)$$
$$\ge(1-\Phi)*(1-\Phi)>1-\theta$$

As $\theta>0$ is arbitrary, so by distribution function,

$$\Im_{\xi_1-\xi_2}(\varepsilon)=1,\qquad\forall\,\varepsilon>0$$
$$\Rightarrow\ \xi_1=\xi_2\ .$$

Hence, uniqueness of the St. $\Im_\Lambda^\alpha$ – limit is proved.

Theorem 3.2 Let $(\mathbb{X};\Im,*)$ be a PN–space and $\alpha\in(0,1]$. If a sequence $x=(x_m)_{m=0}^{\infty}$ is Λ- convergent $\xi$with respect to $\Im$, then St. $\Im_\Lambda^\alpha-\lim x_m=\xi$. But converse may be not true.

Proof: As we know, for every $\theta\in(0,1)$ and $\varepsilon>0$ there exists a number $m_0\in N$ such that,

$\Im(\Lambda x_m-\xi)(\varepsilon)>1-\theta$, whenever $m\ge m_0$.

$\Rightarrow$ The set $\{m\in N:\Im(\Lambda x_m-\xi)(\varepsilon)\le 1-\theta\}$ has almost finitely many terms.

This implies that the α – density of finite set is also zero.

$\therefore\ \delta_{\Lambda^\alpha}(\{m\in N:\Im(\Lambda x_m-\xi)(\varepsilon)\le 1-\theta\})=0$.

Thus, $\mathrm{St}\Im_\Lambda^\alpha-\lim x_m=\xi$.

Now, the next example shows the contrary part of above mentioned theorem is not true.

Example 3.1 Consider $(\mathbb{X},|.|)$ is a real space under the usual norm. Let $p*q=pq$ and $\Im(\Lambda x_m)(v)=\frac{v}{v+|\Lambda x_m|}$, provided $v\ne 0$.

Here, it is noted that $(\mathbb{X};\Im,*)$ be a PN-space. Consider a sequence whose terms are

$$\Lambda x_m=\begin{cases}1, & (m=r^2;r\in Z^+)\\ 0, & \text{otherwise}\end{cases}$$

Then for all $\Phi\in(0,1)$ and any $\varepsilon>0$, let

$$M_{m_0}(\Phi,\varepsilon)=\left\{m\le m_0:\frac{v}{v+|\Lambda x_m|}\le 1-\Phi\right\}$$
$$=\left\{m\le m_0:|\Lambda x_m|\ge\frac{\Phi v}{1-\Phi}>0\right\}$$
$$=\{m\le m_0:|\Lambda x_m=1|\}$$
$$=\{m\le m_0:m=r^2,\ r\in Z^+\},$$

We have,

$$\frac{1}{{m_0}^\alpha}\left|M_{m_0}(\Phi,\varepsilon)\right|=\frac{1}{{m_0}^\alpha}|\{m\le m_0:m=r^2,\ r\in Z^+\}|$$
$$\le\frac{\sqrt{m_0}}{{m_0}^\alpha}\ ,$$

$\Rightarrow\ \lim_{m_0\to\infty}\frac{1}{{m_0}^\alpha}\left|M_{m_0}(\Phi,\varepsilon)\right|=0$ when $\alpha\in\left(\frac{1}{2},1\right]$

$\Rightarrow$ $\mathrm{St}\Im_\Lambda^\alpha-\lim x_m=0$.

Then, Λx_m is not convergent in the space $(\mathbb{X},|.|)$.

Theorem 3.3 Let $(\mathbb{X};\Im,*)$ be a PN–space and $\alpha \in (0,1]$. A sequence $x=(x_m)_{m=0}^{\infty}$ is statistically Λ^α -convergent if and only if it is statistically Λ^α – Cauchy.

Proof: Let $\mathrm{St}\Im_\Lambda^\alpha - \lim x_m = \xi$.

Now, for all $\Phi \in (0,1)$ and any $\varepsilon > 0$, we have

$$\delta_{\Lambda^\alpha}(\{m \in N: \Im(\Lambda x_m - \xi)(\varepsilon) \le 1-\Phi\}) = 0.$$

Now, by choosing a number $m_0 \in N$ such that m and $n \ge m_0$

$$A(\Phi,\varepsilon) = \{m \in N: \Im(\Lambda x_m - \Lambda x_n)(\varepsilon) \le 1-\Phi\},$$
$$B(\Phi,\varepsilon) = \{m \in N: \Im(\Lambda x_m - \xi)(\varepsilon) \le 1-\Phi\},$$
$$C(\Phi,\varepsilon) = \{n \in N: \Im(\Lambda x_n - \xi)(\varepsilon) \le 1-\Phi\}.$$

Then, $A(\Phi,\varepsilon) \subseteq B(\Phi,\varepsilon) \cup C(\Phi,\varepsilon)$.

Therefore,

$$\delta_{\Lambda^\alpha}\big(A(\Phi,\varepsilon)\big) \le \delta_{\Lambda^\alpha}\big(B(\Phi,\varepsilon)\big) + \delta_{\Lambda^\alpha}\big(C(\Phi,\varepsilon)\big).$$

Hence $(x_m)_{m=0}^{\infty}$ is a Λ^α – Cauchy.

Conversely,

Suppose that the sequence $x=(x_m)_{m=0}^{\infty}$ is statistically Λ^α–Cauchy, but not statistically Λ^α– convergent.

Then, for $\varepsilon > 0$ and $\Phi \in (0,1)$, there exists $m_0 \in N$ such that the set $E(\Phi,\varepsilon) = \{m \in N: \Im(\Lambda x_m - \Lambda x_{m_0})(\varepsilon) \le 1-\Phi\}$, for all $m \ge m_0$ has natural density zero i.e. $\delta_{\Lambda^\alpha}(E(\Phi,\varepsilon)) = 0$.

$\Rightarrow \Im(\Lambda x_m - \Lambda x_{m_0})(\varepsilon) \le 2\,\Im(\Lambda x_m - \xi) < 1-\Phi$, if $\Im(\Lambda x_m - \xi) < \frac{1-\Phi}{2}$.

Since $x=(x_m)_{m=0}^{\infty}$ is not statistically Λ – convergent, then

$$\delta_{\Lambda^\alpha}(\{m \in N: \Im(\Lambda x_m - \xi)(\varepsilon) > 1-\Phi\}) = 0$$
$$\Rightarrow \delta_{\Lambda^\alpha}\big(\{m \in N: \Im(\Lambda x_m - \Lambda x_{m_0})(\varepsilon) > 1-\Phi\}\big) = 0.$$

which is a contradiction that $E^c(\Phi,\varepsilon)$ has natural density 1 as $\delta_{\Lambda^\alpha}(E(\Phi,\varepsilon)) = 0$.

Thus, sequence $x=(x_m)_{m=0}^{\infty}$ is statistical Λ^α – convergent.

Theorem 3.4 Let $(\mathbb{X};\Im,*)$ be a PN–space. If $\mathrm{St}\Im_\Lambda^\alpha - \lim x_m = \xi_1$ and $\mathrm{St}\Im_\Lambda^\alpha - \lim y_m = \xi_2$ then,

(i) $\mathrm{St}\Im_\Lambda^\alpha - \lim (x_m + y_m) = \xi_1 + \xi_2$,

(ii) $\mathrm{St}\Im_\Lambda^\alpha - \lim (\beta y_m) = \beta\xi_1$ where $\beta \in \mathcal{R}$.

Proof: (i) As $\mathrm{St}\Im_\Lambda^\alpha - \lim x_m = \xi_1$ and $\mathrm{St}\Im_\Lambda^\alpha - \lim y_m = \xi_2$.

Let $\varepsilon > 0$ and $\theta \in (0,1)$, choose $\Phi \in (0,1)$ such that $(1-\Phi)*(1-\Phi) > 1-\theta$.

Take, $M_1(\Phi,\varepsilon) = \{m \in N: \Im(\Lambda x_m - \xi_1)(\varepsilon) \le 1-\Phi\}$,

$M_2(\Phi,\varepsilon) = \{m \in N: \Im(\Lambda y_m - \xi_2)(\varepsilon) \le 1-\Phi\}$.

Since, $\mathrm{St}\Im_\Lambda^\alpha - \lim x_m = \xi_1 \Rightarrow \delta_{\Lambda^\alpha}\{M_1(\Phi,\varepsilon)\} = 0$, $\forall\ \varepsilon > 0$.

By using, $\mathrm{St}\Im_\Lambda^\alpha - \lim y_m = \xi_2$, we get $\delta_{\Lambda^\alpha}\{M_2(\Phi,\varepsilon)\} = 0$,$\forall\ \varepsilon > 0$.

As $M(\Phi,\varepsilon) = M_1(\Phi,\varepsilon) \cap M_2(\Phi,\varepsilon) \Rightarrow \{M(\Phi,\varepsilon)\} = 0$,

$$\Rightarrow \delta_{\Lambda^\alpha}(\{m \in N: \Im(\Lambda x_m - \xi)(\varepsilon) > 1-\Phi\}) = 1.$$

Then,

$$\begin{aligned}\Im(\Lambda^\alpha x_m + \Lambda^\alpha y_m) - (\xi_1+\xi_2)(\varepsilon) &= \Im(\Lambda x_m - \xi_1 + \Lambda y_m - \xi_2)(\varepsilon)\\ &= \Im(\Lambda x_m - \xi_1)\left(\frac{\varepsilon}{2}\right) * \Im(\Lambda y_m - \xi_2)\left(\frac{\varepsilon}{2}\right)\\ &> (1-\Phi)*(1-\Phi)\\ &> 1-\theta\end{aligned}$$

This shows that

$\delta_{\Lambda^\alpha}(\{m \in N: \Im(\Lambda x_m - \xi_1 + \Lambda y_m - \xi_2)(\varepsilon) \le 1-\theta\}) = 0$.

Therefore, $\mathrm{St}\Im_\Lambda^\alpha - \lim (x_m + y_m) = \xi_1 + \xi_2$.

(ii) Let $\mathrm{St}\Im_\Lambda^\alpha - \lim x_m = \xi_1$. Then for $\theta \in (0,1)$ and $\varepsilon > 0$,

Suppose $\beta = 0$.

Then, $\Im(0*\Lambda x_m - 0*\xi_1)(\varepsilon) = 1 > 1-\theta$.

So, $\Im(0*\Lambda x_m) = 0$.

Now,

Let $\beta \in \mathcal{R}$,$(\beta \neq 0)$.

Since, $\delta_{\Lambda^\alpha}\big(M(\Phi,\varepsilon)\big) = 0$ as St. $\Im_\Lambda^\alpha - \lim = \xi_1$.

where $M(\Phi,\varepsilon) = \{m \in N: \Im(\Lambda x_m - \xi_1)(\varepsilon) \le 1-\theta\}$.

Also, $\delta_{\Lambda^\alpha}(\{m \in N: \Im(\Lambda x_m - \xi_1)(\varepsilon) > 1-\theta\}) = 1$.

For $m \in N - M(\Phi,\varepsilon)$, we have

$$\begin{aligned}\Im(\Lambda\beta x_m - \beta\xi_1)(\varepsilon) &= \Im(\Lambda x_m - \xi_1)\left(\frac{\varepsilon}{|\beta|}\right)\\ &\ge \Im(\Lambda x_m - \xi_1)(\varepsilon) * \Im(0)\left(\frac{\varepsilon}{|\beta|} - \varepsilon\right)\\ &= \Im(\Lambda x_m - \xi_1)(\varepsilon) * 1\\ &= \Im(\Lambda x_m - \xi_1)(\varepsilon)\\ &> 1-\theta\end{aligned}$$

It gives,

$$\delta_{\Lambda^\alpha}(\{m \in N: \Im(\beta\Lambda x_m - \beta\xi_1)(\varepsilon) \le 1-\theta\}) = 0.$$

$\Rightarrow \mathrm{St}\Im_\Lambda^\alpha - \lim (\beta x_m) = \beta\xi_1$.

Theorem 3.5 Let $(\mathbb{X};\Im,*)$ be a PN–space and $\alpha \in (0,1]$. Then St. $\Im_\Lambda^\alpha - \lim x_m = \xi$ iff there exists a set $M = \{m_1 < m_2 < m_3 < \cdots < m_n < \cdots\} \subseteq N$ with $\delta_{\Lambda^\alpha}(M) = 1$ such that $\mathrm{St}\Im_\Lambda^\alpha - \lim x_{m_n} = \xi$.

Proof: Since $\mathrm{St}\Im_\Lambda^\alpha - \lim x_m = \xi$, then for each $\varepsilon > 0$ and $p \in N$. Define

$$M(p,\varepsilon) = \left\{m \in N: \Im(\Lambda x_m - \xi)(\varepsilon) \le 1-\frac{1}{p}\right\}$$

and $A(p,\varepsilon) = \left\{m \in N: \Im(\Lambda x_m - \xi)(\varepsilon) > 1-\frac{1}{p}\right\}$.

Then,$\delta_{\Lambda^\alpha}\big(M(p,\varepsilon)\big) = 0 \Rightarrow \delta_{\Lambda^\alpha}\big(A(p,\varepsilon)\big) = 1$ (3.1)

and

$A(1,\varepsilon) \supset A(2,\varepsilon) \supset \cdots \supset A(j,\varepsilon) \supset A(j+1,\varepsilon) \supset$ (3.2)

Now we shall show that for $m \in M(p,\varepsilon)$ the sequence $x=(x_m)$ is $\mathrm{St}\Im_\Lambda^\alpha - \lim x_m = \xi$. Therefore, there exists $\Phi \in (0,1)$for $\varepsilon > 0$ such that the set $\{m \in N: \Im(\Lambda x_m - \xi)(\varepsilon) \le 1-\Phi\}$ has infinitely many elements.

Assume $A(\Phi,\varepsilon) = \{m \in N: \Im(\Lambda x_m - \xi)(\varepsilon) > 1-\Phi\}$ then $\delta_{\Lambda^\alpha}\big(A(\Phi,\varepsilon)\big) = 0$.

For $\Phi > \frac{1}{p}$, $(p \in N)$ we get $A(p,\varepsilon) \subset A(\Phi,\varepsilon)$ using (3.2).

Hence, $\delta_{\Lambda^\alpha}\big(A(p,\varepsilon)\big) = 0$, which contradict to (3.1).

Therefore, $\mathrm{St}\Im_\Lambda^\alpha - \lim x_{m_n} = \xi$.

Sufficient part:

Suppose there is a subset $M = \{m_1 < m_2 < m_3 < \cdots < m_n < \cdots\} \subseteq N$ with $\delta_{\Lambda^\alpha}(M) = 1$ such that $\Im_\Lambda^\alpha \lim_{m\to\infty} x_{m_n} = \xi$. Then for every $\Phi \in (0,1)$ and $\varepsilon > 0$,there exists $A_I \in N$ such that $\Im(\Lambda x_m - \xi)(\varepsilon) > 1-\Phi$for all $m \ge A_I$.

Then,

$$\begin{aligned}A(\Phi,\varepsilon) &= \{m \in N: \Im(\Lambda x_m - \xi)(\varepsilon) \le 1-\Phi\}\\ &\subseteq N - \{m_{A_{I+1}}, m_{A_{I+2}}, \ldots\ldots\ldots\}\\ &\Rightarrow \delta_{\Lambda^\alpha}\big(A(\Phi,\varepsilon)\big) \le 1-1 = 0\end{aligned}$$

Therefore, $\mathrm{St}\Im_\Lambda^\alpha - \lim x_m = \xi$.

IV. CONCLUSIONS

This particular paper has taken the concept of statistically Λ–convergent and studied its generalization with order α in the environment of PN–spaces. Also we have given some illustrative examples to reveal the concepts.

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CONFLICT OF INTEREST

Authors have no any conflict of interest.

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