



## Numerical Solution of Instabilities in Displacement Through Porous Media Using Fractional Calculus

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**ABSTRACT :** In this paper, by introducing the fractional derivative in caputo sense, we apply the Adomain decomposition method. The application of Adomain decomposition method, developed for differential equations of integer order. The solution of our model equation is calculated in the form of convergent series with easily computable contents. The example of fluid phenomenon in instabilities in displacement through porous media with is presented to show the application of the present technique. Numerical results show that the approach of Adomain decomposition method is easy to implement and accurate when applied to partial differential equations.

**Keywords :** Fractional Diffusion-wave equation, Caputo fractional derivative, A domain decomposition method.

### I. INTRODUCTION

The unsteady and unsaturated flow of water through soils is due to content changes as a function of time and entire pore spaces are not completely filled with flowing liquid respectively. Knowledge concerning such flows some helps some workers like hydrologist, agriculturalists, many fields of science and engineering. The water infiltrations system and the underground disposal of seepage and waste water are encountered by these flows, which are described by nonlinear partial differential equation.

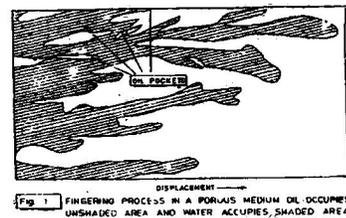
The mathematical model conforms to the hydrological solution of one dimensional vertical ground water recharge by spreading. Such flow is of great importance in water resource science, soil engineering and agricultural sciences.

If a fluid contained in a porous medium is displaced by another fluid of lesser viscosity, then it is frequently observed that the displacing fluid has a strong tendency to protrude in form of fingers (instabilities) into more viscous fluid. This phenomenon is called fingering.

In Fig. 1, fingering process has been shown between oil-water flows into a porous medium. In petroleum engineering, the fingering process is a well-known phenomenon occurring in displacement of oil by water by flooding that is a common oil recovery technique.

In the statistical treatment of fingering [8] only average cross-sectional area occupied by the fingers was observed while the size and shape of the individual fingers are neglected as in Fig. 2. Scheidegger and Jhonson [9] discussed the statistical behaviour in homogeneous porous media with capillary pressure. Verma [10] has examined the behaviour of fingering in a displacement process through heterogeneous porous media.

In this paper, we have numerically discussed the phenomenon of instabilities in a displacement process involving two immiscible liquids. Numerical solution of governing non-linear partial differential equation has been obtained by ADM. The numerical results are obtained at various time levels.



### II. DESCRIPTION OF FRACTIONAL CALCULUS

There are mathematical definitions about fractional derivative [1, 2]. Here, we adopt the two usually used definitions: the caputo and its reverse operator Riemann-Liouville. That is because caputo fractional derivative allows traditional initial condition assumption and boundary conditions. More details one can consult [1]. We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

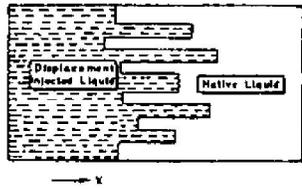


Fig 2 SCHEMATIC REPRESENTATION OF FINGERS AT LEVEL 'x'.

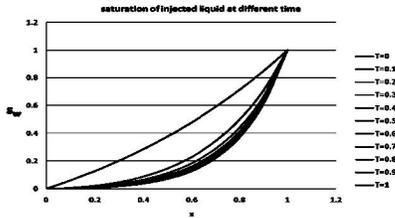


Fig. 3. Saturation of injected liquid at different time.

**Definition 1:**

The Reimann-Liouville fractional integral operator of order  $\alpha \geq 0$  is defined as,

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt; \quad \alpha > t, x > 0$$

$$J^0 f(x) = f(x)$$

The properties of  $J^\alpha$  are  $J^\alpha J^\beta = J^{\alpha+\beta}$  for  $\alpha, \beta \geq 0$  which implies the commutative property  $J^\beta J^\alpha = J^{\alpha+\beta}$ .

For  $F \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$  and  $\gamma - 1$

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma)}{\Gamma(\gamma+1+\alpha)} t^{\gamma+\alpha}$$

The Reimann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce now a modified fractional differential operator  $D_*^\alpha$  proposed by M. Caputo in his work on the theory of viscoelasticity.

**Definition 2:**

The fractional derivative of  $f(x)$  in the Caputo sense is defined as,

$$\begin{aligned} D_*^\alpha f(x) &= D^\alpha J^{m-\alpha} f(x) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^m(t) dt \end{aligned}$$

For  $m - 1 < \alpha \leq 1, m \in N$  and  $F \in C_\mu^m, \mu \geq -1$  then,  $D_*^\alpha J^\alpha f(x)$

$$= f(x) \text{ and } J^n D_*^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x^k}{k!}, x > 0$$

In this paper, the fractional derivatives are considered in the Caputo sense. The reason for adopting the Caputo definition is as follows:

To solve differential equation (both classical and fractional), we need to specify additional conditions in order to produce a unique solution. For the case of Caputo fractional differential equations, these additional conditions are just the traditional conditions, which are akin to those of classical differential equations and are therefore familiar to us. In contrast, for Riemann-Liouville fractional differential equations, these additional conditions constitute certain fractional derivatives (and/or integrals) of the unknown solution at the initial point  $x = 0$ , which are functions of  $x$ . These initial conditions are not physical. Furthermore it is not clear how such qualities are to be measured from experiment, say, so that they can be appropriately assigned in an analysis [10].

**III. ANALYSIS OF THE NUMERICAL METHOD**

We consider the following time-fractional partial differential equation

$$D_{*t}^\alpha = f(u, u_x, u_{xx}) + g(x, t), \quad m - 1 < \alpha \leq m \quad \dots (1)$$

where  $D_{*t}^\alpha$  is the Caputo fractional derivative of order  $\alpha, m \in N, f$  is a nonlinear function and  $g$  is the source function.

The decomposition method requires that the nonlinear fractional differential equation (1) be expressed in terms of operator from as

$$D_{*t}^\alpha u(x, t) + Lu(x, t) + Nu(x, t) = g(x, t), \quad x > 0 \quad \dots (2)$$

where  $L$  is a linear operator which might include other fractional derivatives of order less than  $\alpha$ ,  $N$  is a non-linear operator which also might include other fractional derivatives of order less than  $\alpha$ ,  $g(x, t)$  and  $D_{*t}^\alpha$  are defined as in equation (1).

Applying the operator  $J^\alpha$ , the inverse of the operator  $D_{*t}^\alpha$ , to both sides of equation (1) yields

$$\begin{aligned} u(x, t) &= \sum_{k=0}^{m-1} \frac{\partial^k u}{\partial t^k} (x, 0^+) \frac{t^k}{k!} + J^\alpha g(x, t) \\ &\quad - J^\alpha [Lu(x, t) + Nu(x, t)] \end{aligned} \quad \dots (3)$$

The Adomian decomposition method [1-4] suggests the solution  $u(x, t)$  be decomposed into the infinite series of components

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) \quad \dots (4)$$

and the nonlinear function in equation is decomposed

as follows:

$$Nu = \sum_{n=0}^{\infty} A_n \quad \dots (5)$$

where  $A_n$  are so-called the Adomian polynomials.

Substitution the decomposition series (4) and (5) into both sides of (3) gives

$$u(x, t) = \sum_{k=0}^{m-1} \frac{\partial^k u}{\partial t^k}(x, 0^+) \frac{t^k}{k!} + j^\alpha g(x, t) - J^\alpha \left[ L \left( \sum_{n=0}^{\infty} u_n(x, t) + \sum_{n=0}^{\infty} A_n \right) \right] \quad \dots (6)$$

From this equation, the iterates are determined by the following recursive way

$$\begin{aligned} u_0(x, t) &= \sum_{k=0}^{m-1} \frac{\partial^k u}{\partial t^k}(x, 0^+) \frac{t^k}{k!} + j^\alpha g(x, t) \\ u_1(x, t) &= -J^\alpha(Lu_0 + A_0) \\ u_2(x, t) &= -J^\alpha(Lu_1 + A_1) \\ &\vdots \\ u_{n+1}(x, t) &= -J^\alpha(Lu_n + A_n) \end{aligned} \quad \dots (7)$$

The Adomian polynomial  $A_n$  can be calculated for all forms of nonlinearity according to specific algorithms constructed by Adomian [3]. The general form of formula for  $A_n$  Adomian polynomials is

$$A_n = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} N \left( \sum_{k=0}^n \lambda^k u_k \right) \right]_{\lambda=0} \quad \dots (8)$$

This formula is easy to compute to get as many polynomials as we need in the calculation of the numerical as well as explicit solutions.

Finally, we approximate the solution  $u(x, t)$  by the truncated series

$$\varnothing_N(x, t) = \sum_{n=0}^{N-1} u_n(x, t)$$

$$\text{and} \quad \lim_{N \rightarrow \infty} \varnothing_N(x, t) = u(x, t) \quad \dots (9)$$

However, in many cases the exact solution in a closed form may be obtained. Moreover, the decomposition series solutions generally converge very rapidly. The convergence of the decomposition series has been investigated in [5-7]. They obtained some results about the speed of convergence of this method. In recent work of Abbaoui and Cherruault [7] have proposed a new approach of convergence of the decomposition series. The authors have given a new condition for obtaining convergence of the decomposition series to the classical presentation of the ADM in [7].

## IV. FINGERING IN A HOMOGENEOUS MEDIUM

### A. Statement of the problem

We consider that there is a uniform water injection into an oil saturated porous medium of homogeneous physical characteristics, such that the injecting water cuts through the oil formation and give rise to protuberance. This furnishes a well developed fingers flow. Since the entire oil at the initial boundary ( $x = 0$ ) is displaced through a small distance due to the water injection. Therefore, we assume, further that complete water saturation exists at the initial boundary.

### B. Formulation of the problem

The seepage of water ( $v_w$ ) and oil ( $v_o$ ) are given by Darcy's Law,

$$v_w = - \left( \frac{k_w}{\delta_w} \right) K \left[ \frac{\partial P_w}{\partial x} \right] \quad \dots (10)$$

$$v_o = - \left( \frac{k_o}{\delta_o} \right) K \left[ \frac{\partial P_o}{\partial x} \right] \quad \dots (11)$$

where  $K$  is the permeability of the homogeneous medium,  $K_w$  and  $K_o$  are relative permeability of water and oil, which are functions of  $S_w$  and  $S_o$  ( $S_w$  and  $S_o$  are the saturation of water and oil) respectively,  $P_w$  and  $P_o$  are pressure of water and oil,  $\delta_w$  and  $\delta_o$  are constant kinematic viscosities,  $\alpha$  is the inclination of the bed and  $g$  is acceleration due to gravity.

Regarding the phase densities are constant, the equations of continuity of the two phase are:

$$P \left( \frac{\partial S_w}{\partial t} \right) + \frac{\partial V_w}{\partial x} = 0 \quad \dots (12)$$

$$P \left( \frac{\partial S_o}{\partial t} \right) + \frac{\partial V_o}{\partial x} = 0 \quad \dots (13)$$

where  $P$  is porosity of the medium. From the definition of phase saturation, it is evident that,

$$S_w + S_o = 1 \quad \dots (14)$$

The capillary pressure  $P_c$  is defined as

$$P_c = -\beta_0 S_w \quad \dots (15)$$

$$P_c = P_o - P_w \quad \dots (16)$$

where  $\beta_0$  is a constant quantity.

At this state, for definiteness of mathematical analysis, we assume standard relationship due to Scheidegger and Jhonson [11], between phase saturation and relative permeability as

$$K_w = S_w \quad \dots (17)$$

$$K_o = S_o = 1 - S_w \quad \dots (18)$$

The equation of motion for saturation can be obtained by substituting the values of  $V_w$  and  $V_o$  from equation (10) and (11) into the equation (12) and (13) respectively, we get,

$$P \left( \frac{\partial s_w}{\partial t} \right) = \frac{\partial}{\partial x} \left[ \left( \frac{K_w}{\delta_w} \right) K \left( \frac{\partial P_w}{\partial x} \right) \right] \quad \dots (19)$$

$$P \left( \frac{\partial s_o}{\partial t} \right) = \frac{\partial}{\partial x} \left[ \left( \frac{K_o}{\delta_o} \right) K \left( \frac{\partial P_o}{\partial x} \right) \right] \quad \dots (20)$$

These are the general flow equations of the phase in homogeneous medium, when effects due to pressure discontinuity and gravity term in inclined porous medium are considered.

Eliminating  $\left( \frac{\partial P_w}{\partial x} \right)$  from equations (19) and (16), we obtain

$$P \left( \frac{\partial s_w}{\partial t} \right) = \frac{\partial}{\partial x} \left[ \left( \frac{K_w}{\delta_w} \right) K \left( \frac{\partial P_o}{\partial x} - \frac{\partial P_c}{\partial x} \right) \right] \quad \dots (21)$$

Combining equation (20) and (21) and using equation (5), we get,

$$\frac{\partial}{\partial x} \left[ K \left( \frac{K_w}{\delta_w} + \frac{K_o}{\delta_o} \right) \frac{\partial P_o}{\partial x} - K \frac{K_w}{\delta_w} \frac{\partial P_c}{\partial x} \right] = 0$$

Integrating above equation with respect to  $x$ , we have

$$K \left( \frac{K_w}{\delta_w} + \frac{K_o}{\delta_o} \right) \frac{\partial P_o}{\partial x} - K \frac{K_w}{\delta_w} \frac{\partial P_c}{\partial x} = -V \quad \dots (22)$$

where  $V$  is constant of integrating which can be evaluated from later on. Simplification of (22) gives

$$\frac{\partial P_o}{\partial x} = \frac{-V}{K \left( \frac{K_w}{\delta_w} + \frac{K_o}{\delta_o} \right)} + \frac{\frac{\partial P_c}{\partial x}}{1 + \left( \frac{K_o}{K_w} \right) \left( \frac{\delta_w}{\delta_o} \right)}$$

Using above equation in (21), we have

$$P \left( \frac{\partial s_w}{\partial t} \right) + \frac{\partial}{\partial x} \left[ \frac{-V}{K \left( \frac{K_w}{\delta_w} + \frac{K_o}{\delta_o} \right)} + \frac{\frac{\partial P_c}{\partial x}}{1 + \left( \frac{K_o}{K_w} \right) \left( \frac{\delta_w}{\delta_o} \right)} \right] = 0 \quad \dots (23)$$

The value of pressure of oil ( $P_o$ ) can be written as [12]

of the form

$$P_o = \frac{P_o + P_w}{2} + \frac{P_o - P_w}{2} = \bar{P} + \frac{1}{2} P_c \quad \dots (24)$$

where  $\bar{P}$  is the mean pressure which is constant, therefore (24) implies

$$\frac{\partial P_o}{\partial x} = \frac{1}{2} \frac{\partial P_c}{\partial x}$$

Using above equation in (22), we get,

$$V = \frac{K}{2} \left[ \left( \frac{K_w}{\delta_w} \right) - \left( \frac{K_o}{\delta_o} \right) \frac{\partial P_c}{\partial x} \right]$$

Substituting the value of  $V$  from above equation in equation (23), we get,

$$P \left( \frac{\partial s_w}{\partial t} \right) + \frac{1}{2} \frac{\partial}{\partial x} \left[ K \left( \frac{K_w}{\delta_w} \right) \left( \frac{\partial P_c}{\partial x} \right) \left( \frac{\partial s_w}{\partial x} \right) \right]$$

Substituting the value of  $K_w$  and  $P_c$  from (15) and (17), we get,

$$P \left( \frac{\partial s_w}{\partial t} \right) - \frac{\beta_o K}{2 s_w} \frac{\partial}{\partial x} \left[ s_w \frac{\partial s_w}{\partial x} \right] = 0 \quad \dots (25)$$

A set of suitable boundary conditions associated to problem (25) are

$$s_w(0, t) = 1; s_w(x, 0) = 0; s_w(L, t) = 0 \quad \dots (26)$$

Equation (25) is reduced to dimensionless form by setting

$$X = \frac{x}{L}, T = \frac{K \beta_o t}{2 \delta_w L^2 P}, s_w(x, t) = s_w^*(x, t)$$

So that  $\frac{\partial s_w}{\partial t} = \frac{\partial}{\partial x} \left( s_w \frac{\partial s_w}{\partial x} \right) \quad \dots (27)$

In equation (27) and (28) the asterisk are dropped for simplicity.

Equation (27) is desired nonlinear differential equation of motion for the flow of immiscible liquid in homogeneous medium.

The problem is solved by using Adomain Decomposition Method. The numerical values are shown by table. Curves indicate the behaviour of saturation of water corresponding to various time periods.

**C. Solution of the problem using Adomain Decomposition Method:**

$$\frac{\partial s_w}{\partial T} = \frac{\partial}{\partial x} \left( s_w \frac{\partial s_w}{\partial x} \right)$$

$$\therefore (s_w)_t = [s_w(s_w)_x]_x \quad \dots (28)$$

Taking the initial condition  $s_w(x, 0) = s_{w0} = f(x)$

Applying the operator  $J$  on both the sides of equation (1) using initial condition,

$$s_w(x, t) = f(x) + J[\Phi_1\{s_w(x, T)\}] + J[\Phi_2\{s_w(x, T)\}] \quad \dots (29)$$

where  $\Phi_1[s_w(x, T)] = s_w(s_w)_{xx}$  and  $\Phi_2[s_w(x, T)] = (s_w)_x^2$

Following Adomain decomposition method, the solution is represented as infinite series like,

$$s_w(x, t) = \sum_{n=0}^{\infty} s_{wn}(x, T) \quad \dots (31)$$

The nonlinear operator  $\Phi_1(s_w)$  and  $\Phi_2(s_w)$  are decomposed in these forms,

$$\Phi_1[s_w(x, t)] = \sum_{n=0}^{\infty} A_n, \Phi_2[s_w(x, t)] = \sum_{n=0}^{\infty} B_n \quad \dots (4)$$

where  $A_n$  and  $B_n$  are so called Adomain polynomials and have the form,

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ \Phi_1 \left( \sum_{k=0}^{\infty} \lambda^k s_{wk} \right) \right]_{\lambda=0}$$

$$= \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ \left( \sum_{k=0}^{\infty} \lambda^k s_{wk} \right) \left( \sum_{k=0}^{\infty} \lambda^k s_{wkxx} \right) \right]_{\lambda=0}$$

$$B_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ \Phi_2 \left( \sum_{k=0}^{\infty} \lambda^k s_{wk} \right) \right]_{\lambda=0}$$

$$= \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ \left( \sum_{k=0}^{\infty} \lambda^k (s_{wkx})^2 \right) \right]_{\lambda=0}$$

The first three components of these polynomials are,

$$A_0 = s_{w0}(s_{w0})_{xx}$$

$$A_1 = s_{w0}s_{w1xx} + s_{w1}s_{w0xx}$$

$$A_2 = s_{w2}s_{w0xx} + s_{w1}s_{w1xx} + s_{w0}s_{w2xx}$$

$$A_3 = s_{w3}s_{w0xx} + s_{w1}s_{w2xx} + s_{w2}s_{w1xx} + s_{w0}s_{w3xx}$$

Similarly,

$$B_0 = (s_{w0x})^2$$

$$B_1 = (s_{w1x})^2$$

$$B_2 = (s_{w2x})^2$$

$$B_3 = (s_{w3x})^2$$

Other polynomials can be generated in like manner, substituting the decomposition series (30) and (31) into equation (29) yields the following recursive formula,

$$s_{w0}(x, t) = f(x)$$

$$s_{wn+1}(x, T) = J(A_n) + J(B_n); n \geq 0$$

Let

$$S_{w0}(x, T) = f(x) = \frac{e^x - 1}{e - 1}$$

$$s_{w1} = J(A_0) + J(B_0)$$

$$= J[s_{w0}(s_{w0})_{xx}] + J[(s_{w0x})^2]$$

$$= f_1 T \text{ where } f_1 = ff_{xx} + f_x^2 = \frac{2e^{2x} - e^x}{(e-1)^2}$$

$$s_{w2} = J(A_1) + J(B_1)$$

$$J(s_{w0}s_{w1xx} + s_{w1}s_{w0xx}) + J[(s_{w1x})^2]$$

$$= f_2 T \text{ where } f_2 = ff_{1xx} + f_1 f_{1x} + f_{1x}^2$$

$$= \frac{10e^{3x+1} - 10e^{2x+1} + e^{x+1} - 10e^{3x} + 11e^{2x} - e^x + 16e^{4x}}{(e-1)^4}$$

$$\therefore s_w(x, t)$$

$$= \frac{e^x - 1}{e - 1} + \frac{2e^{2x} - e^x}{(e-1)^2} T$$

$$+ \frac{10e^{3x+1} - 10e^{2x+1} + e^{x+1} - 18e^{3x} + 11e^{2x} - e^x + 16e^{4x}}{(e-1)^4} \frac{T^2}{2} + \dots$$

### D. Interpretation

In graph X-axis represents the vales of  $x$  and Y-axis represents the saturation of injected liquid ( $s_w$ ) of porous media of length one.

Initially saturation of injected liquid is zero at each point on observed region. Also, there is full injected liquid saturation (*i.e.*  $s_w = 1$ ) at injected face  $x = 0$  and there is no saturation of injected liquid at other end ( $x = 1$ ) inspective of time.

It is clear from graph that, for each value of  $T$ , saturation  $s_w$  has a decreasing tendency along the space co-ordinate axis. Also, for each point of  $X$  saturation increases as time increases but the rate at which it rises at each point in observed region slows down with increase in time. This shows that the stabilization of the fingers is truly possible with the assumption made for capillary pressure and water saturation.

**Table1: The approximate solution for saturation of injected liquid for different values of  $x$  at different time using ADM.**

$x$	$T = 0$	$T = 0.1$	$T = 0.2$	$T = 0.3$	$T = 0.4$
$T = 0.5$	$T = 0.6$	$T = 0.7$	$T = 0.8$	$T = 0.9$	$T = 1$
0	0	0	0	0	0
0	0	0	0	0	0

0											
0.1	0.061207	0.006005	0.004874	0.004118	0.003643	0.003329	0.00311	0.002949	0.002826	0.00273	0.002652
0.2	0.128851	0.021126	0.015964	0.013091	0.011439	0.010404	0.009706	0.009209	0.008838	0.008552	0.008326
0.3	0.20361	0.047578	0.035451	0.029066	0.025509	0.023329	0.021882	0.020864	0.020113	0.019538	0.019085
0.4	0.28623	0.088276	0.066492	0.055299	0.049153	0.045425	0.042972	0.041255	0.039996	0.039036	0.038284
0.5	0.377541	0.147133	0.113719	0.096753	0.087505	0.081923	0.078266	0.075715	0.073849	0.072431	0.071321
0.6	0.478454	0.229499	0.184015	0.161045	0.148567	0.141055	0.136143	0.132722	0.130223	0.128326	0.126843
0.7	0.58998	0.342833	0.287697	0.259904	0.244824	0.235753	0.229826	0.225701	0.222688	0.220402	0.218615
0.8	0.713236	0.497707	0.440362	0.411444	0.395749	0.386306	0.380135	0.37584	0.372703	0.370323	0.368462
0.9	0.849455	0.709352	0.665723	0.643681	0.631704	0.624492	0.619775	0.616491	0.61409	0.612269	0.610844
1	1	1	1	1	1	1	1	1	1	1	1

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