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Ideal and Distributive Lattice

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ABSTRACT : In this paper we are giving some important results on Ideal of a Lattice.

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I. INTRODUCTION:

The purpose of this paper is to prove some significant results on Ideal. In section 1 we consider Lattice and Ideal. And in section 2, we think about Distributive and modular Lattice.

Theorem: The ideal kernel of a homomorphism is an ideal of Lattice L.

Proof: $f : L \to L'$

kerf = $\{x : f(x) = 0'\}$

as $f(0) = 0'$

Therefore. Kerf is nonempty set.

i.e. $0 \in$ Kerf now,

first we prove ker $f \subseteq L$.

$0 \in$ ker f $0 \in L \Rightarrow 0 \wedge 0 \in$ ker f

If $a, b \in$ ker f then $f(a) = 0'$ and $f(b) = 0'$

as f is homomorphism therefore

$\Rightarrow f(a \wedge b) = f(a) \wedge f(b)$

$\Rightarrow 0' \wedge 0'$

$\Rightarrow 0'$

$\Rightarrow a \wedge b \in$ ker f

$\Rightarrow$ Therefore kerf is sub lattice of L.

let $a \in L$, $i \in l$

as $i \in I$ therefore $f(i) = 0'$

$f(a \wedge i) = f(a) \wedge f(i)$

$\Rightarrow f(a) \wedge 0'$

$\Rightarrow 0'$

$\Rightarrow a \wedge i$ Î ker f

$\Rightarrow$ Therefore it is clear that kerf is an ideal of lattice L.

Theorem: Every congruence relation of $L \times K$ is of this form in lattice but not true in Abelian group.

Proof: First part is proved in [gratzer]. Now let ψ be a congruence relation on $L \times K$. For $a, b \in L$ define $a \equiv b(\phi)i\ ff\ (a, c) \equiv (b, c)\ (\phi)$ for some $c \in K$. Let d_k.

Joining both sides with $(a \wedge b, d\ )$ and then meeting with $(avb\ d,\ )$, we get $(a, d)\ (b, d\ )\ (\psi)$; thus $(a, c) \equiv (b, c)$ for some $c \equiv K$ is equivalent to $(a, c) \equiv (b, c)$ for all $c \in K$.

Similarly,

define for $a, b \in K$, $a \equiv b\ (\Theta)$ if $f\ (c, a) \equiv (c, b)\ (\psi)$ for all $c \in L$. It is easily seen that ϕ and Θ are congruences. Let $(a, b) \equiv (c, d)\ (\phi \times x)$; then $(a, x) \equiv (c, X)\ (\psi)$, $(y, b) \equiv (y, d)\ (\psi)$, for all $x \in K$ and $y \in L$. Joining the two with $y = a \wedge c$ and $x = b \wedge d$ we get $(a, b)\ (c, d)\ (\psi)$. Finally, let $(a, b)\ (c, a)\ (\psi)$. Meeting with $(a \vee C, b \wedge d)$, we get $(a, b \wedge d)\ (c, b \wedge d)\ (\psi)$; therefore, $a \equiv c\ (\phi)$. Similarly, $b \equiv d\ (\Theta)$,

and so $(a, b) \equiv (c, d)\ (\phi \times \Theta)$

proving that $y \equiv \phi \times \Theta$.

In above we defined $a \leq b$. which implies $a \wedge b = a$ and $a \vee b = b$ But if we consider abelian group then it is not possible.

Theorem: $I(L)$ is complemented lattice iff L has zero.

Proof: Let $I(L)$ is complemented lattice. If $a \in I(L)$ then there exist a' such that $a \wedge a' = 0$ and $a \vee a' = 1$ *i.e.* $I(L)$ has a zero. And as $I(L) \subseteq L$. Therefore L has a zero.

Suppose that L has a zero. *i.e.* $0 \in L$. $I(L)$ is a ideal of L.

$0 \in L$ and $a \in I(L)$ then $0 \wedge a \in I(L)$

$\Rightarrow 0 \in I(L)$

As $I(L)$ is a sub lattice of L. Therefore there exist a and b such that $a \wedge b$, – 0. By duality there exists $1 \in I(L)$. As 0 and $1 \in I(L)$. Therefore there exist $a \vee b$.

Because $o' = 1$ and $1' = 0$. and as $a \wedge b = 0$ and $a \vee b$. therefore $I(L)$ is complemented lattice.

Cor. If $I(L)$ is complimented then it is complete lattice.

Theorem: If L be a relatively complemented lattice, $I, J \in I(L)$, and $I \subseteq J$. , if I is an intersection of prime ideals, then so is J.

Proof: As J is superset then by the definition I and J are ideals.

$I \subseteq J$.

It is given that I is an intersection of prime ideals

i.e. $\wedge (P_1 \cap P_2) = I \subseteq J$

$\Rightarrow P_1 \cap P_2 = J$

Section 2

Theorem: If L is a distributive lattice then so is $I(L)$.

Proof: It is given that L is distributive lattice. We have to prove $I(L)$ is also distributive lattice.

As $I(L)$ is ideal. Therefore $a \wedge i_1 \in I(L), a \in L, i_1 \in l$

$a \wedge i_2 \in I(l), a \in l, i_2 \in l$

As $I(L)$ is a sublattice of L.

Therefore $(a \wedge i_1) V a \wedge i_2 \in I(L)$

$\Rightarrow$ $a \wedge (i_1 \vee i_2) \in I(L)$

$\Rightarrow$ $I(L)$ is a distributive lattice.

Theorem : If L is modular iff $I(L)$ is modular

Proof: If L is modular, then every sub lattice of L is also modular; and as $I(L)$ is a sublattice therefore it is modular. Conversely, let L be non modular,

let $a, b, c \in L$, $a \le b$, and let $(u \wedge c)vb + a \wedge (cvb)$. The free lattice generated by a, b, and c with $a \ge b$. Therefore, the sublattice of L i.e. $I(L)$ generated by a, b, and c must be a homomorphic image of pentagon. If any two of the five elements $a \wedge c$, $(a \wedge c)$, $a \wedge (bvc)$, bvc, c are identified under a homomorphism, then so are $(a \wedge c)vb$ and $a \wedge (bvc)$. Consequently, these five elements are distinct in L, and they form a pentagon.

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