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Common Fixed Point in Hilbert Space by Ishikawa Iterations

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ABSTRACT: In this present paper we have shown that for mappings T_1 and T_2 on Hilbert space satisfying rational condition (10), if the sequence of Ishikawa iterates converges, it converges to common fixed point of mappings T_1 and T_2. Our purpose here is to generalize the result [9, 15].

Keywords and Phrases. Common fixed point, Hilbert space, Contraction mappings.

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I. INTRODUCTION

In recent time there are various iteration methods are applied to fixed point problem of operators. Two general iterations which have been applied are, Mann iteration scheme [7] and Ishikawa scheme [5]. The Mann iteration scheme was introduced by W. Robert Mann [7] and applied to fixed point problems.

Let C be a nonempty convex subset of normed space X. Let $T: C \to C$ be two mappings. The Mann iteration scheme is defined by the sequence $\{x_n\}$ as follows:

$$x_1 = x \in C,$$
$$x_{n+1} = (1-b_n)x_n + b_n T x_n, \quad n \in N$$

where N denotes the set of all positive integer and $\{x_n\}$ is a sequence in [0,1].

Hicks and Kubicek [4], Rhoades [11,12], Singh [16,17] have been shown that for mapping T satisfying certain contractive condition, if the sequence of the Mann iterates converges it converges to a fixed point of T.

The Ishikawa Scheme [5] was introduced by Shiro Ishikawa and it used to establish the strong convergence for a Lipschitzian pseudo-contractive mapping of a compact convex subset E of Hilbert space X into itself.

In the Ishikawa Scheme $\{\alpha_{2n}\}$, $\{\beta_{2n}\}$ satisfies $0 \le \alpha_{2n} \le \beta_{2n} \le 1$ for all n. $\lim \beta_{2n} = 0$ as $n \to \infty$ and $\sum \alpha_{2n}\beta_{2n} = \infty$. In this paper we shall make assumptions that:

(i) $0 \le \alpha_{2n} \le \beta_{2n} \le 1$ for all n,

(ii) $\lim_{n\to\infty} \alpha_{2n} = \alpha_0 > 0.$

(iii) $\lim_{n\to\infty} \beta_{2n} = \beta_0 > 1.$

Let X be a Banach space and let C be a nonempty subset of X. Let T_1 and $T_2 : C \to C$ be two mappings.

The iteration scheme, called I-scheme, is defined as follows:

$$x_0 \in C \quad \text{...(1)}$$

$$\left.\begin{aligned} y_{2n} &= \beta_{2n}T_1x_{2n} + (1-\beta_{2n})x_{2n}, \quad n \ge 0 \\ x_{2n+1} &= (1-\alpha_{2n})x_{2n} + \alpha_{2n}T_2y_{2n}, \quad n \ge 0 \end{aligned}\right\} \quad \text{...(2)}$$

$$\left.\begin{aligned} y_{2n+1} &= \beta_{2n+1}T_1x_{2n+1} + (1-\beta_{2n+1})x_{2n+1}, \quad n \ge 0 \\ x_{2n+2} &= (1-\alpha_{2n+1})x_{2n+1} + \alpha_{2n+1}T_2y_{2n+1}, \quad n \ge 0 \end{aligned}\right\} \quad \text{...(3)}$$

Naimpally and Singh [8] , Kalinde and Rhoades[6] , Rhoades [13] , Sayyed and Badshah [15] shown that or a mapping T or pair of mappings T_1 and T_2 satisfying a contractive condition,if the sequence of Ishikawa iterates converges,it converges to the fixed point of T or pair of mappings T_1 and T_2.

We know that Banach space is Hilbert space if and only if its norm satisfies the parallelogram law, i.e., for every $x, y \in X$.

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \ldots(4)$$

which implies,

$$\|x+y\|^2 \le 2\|x\|^2 + 2\|y\|^2 \quad \ldots(5)$$

we often use this inequality throughout the result.

In 1974, S. Ishikawa [5] proved a certain sequence of points which is iteratively defined converges always to a fixed point of a Lipschitzian pseudo-contractive mapping T:

$$\|Tx - Ty\| \le L\|x-y\| \quad \text{for } x, y \text{ in } E \text{ and } L \text{ is positive number} \quad \ldots(6)$$

We prove the result concerning the existence of common fixed point of pairs of mappings satisfying the contractive condition of the type

$$\|T_1x - T_2y\| \le \left\{\alpha + \beta \frac{\|x - T_1x\|^2}{\left(1 + \|x-y\|^2\right)}\right\}\|y - T_2y\|^2$$

Theorem 1: Let X be a Hilbert space and let C be a closed convex subset or X. Let T_1 and T_2 be two mappings satisfying

$$\|T_1x - T_2y\| \le \left\{\alpha + \beta \frac{\|x - T_1x\|^2}{\left(1 + \|x-y\|^2\right)}\right\}\|y - T_2y\|^2 \quad \ldots(7)$$

Where $\alpha + \beta < 1/4$.

If there exists a point x_0 such that the I- scheme for T_1 and T_2 defined by (2) and (3), converges to a point u, $x_{2n} \to u$, $\|x_{2n+1} - x_{2n}\| \to 0$. then u is a common point of T_1 and T_2 .

Proof: It follows from (2) that $x_{2n+1} - x_{2n} = \alpha_{2n}(T_2\, y_{2n} - x_{2n})$. Since $x_{2n} \to u$, $\|x_{2n+1} - x_{2n}\| \to 0$.

Since $\{\alpha_{2n}\}$ is bounded away from zero, $\|Ty_{2n} - x_{2n}\| \to 0$ as $n \to \infty$. It follows that $\|u - Ty_{2n}\| \to 0$ as $n \to \infty$.

Since T_1 and T_2 satisfy (7) we have,

$$\|T_1x_{2n} - T_2y_{2n}\|^2 \le \left\{\alpha + \beta \frac{\|x_{2n} - T_1x_{2n}\|^2}{\left(1 + \|x_{2n} - y_{2n}\|^2\right)}\right\}\|y_{2n} - T_2y_{2n}\|^2 \quad \ldots(8)$$

Now,

$$\begin{aligned}
\|y_{2n} - x_{2n}\|^2 &= \|\beta_{2n}T_1x_{2n} + (1-\beta_{2n})x_{2n} - x_{2n}\|^2 \\
&= \|\beta_{2n}T_1x_{2n} + x_{2n} - \beta_{2n}x_{2n} - x_{2n}\|^2 \\
&= \|\beta_{2n}(T_1x_{2n} - x_{2n})\|^2 \\
&= \beta_{2n}{}^2\,\|(T_1x_{2n} - T_2y_{2n}) + (T_2y_{2n} - x_{2n})\|^2 \\
&\le 2\|T_1x_{2n} - T_2y_{2n}\|^2 + 2\|(T_2y_{2n} - x_{2n})\|^2 \quad \ldots(9)
\end{aligned}$$

and

$$\begin{aligned}\|y_{2n}-T_2y_{2n}\|^2 &= \|\beta_{2n}T_1x_{2n}+(1-\beta_{2n})x_{2n}-T_2y_{2n}\|^2\\ &= \|\beta_{2n}T_1x_{2n}+(1-\beta_{2n})x_{2n}-T_2y_{2n}+\beta_{2n}T_2y_{2n}-\beta y_{2n}T_2y_{2n}\|^2\\ &= \|\beta_{2n}(T_1x_{2n}-T_2y_{2n})+(1-\beta_{2n})(x_{2n}-T_2y_{2n})\|^2\\ &\le 2\beta^2{}_{2n}\|T_1x_{2n}-T_2y_{2n}\|^2+2(1-\beta_{2n})^2\|x_{2n}-T_2y_{2n}\|^2\\ &\le 2\|T_1x_{2n}-T_2y_{2n}\|^2+2\|x_{2n}-T_2y_{2n}\|^2 \qquad ...(10)\end{aligned}$$

From (9) and (10) , (8) can be written as:

$$\begin{aligned}\|T_1x_{2n}-T_2y_{2n}\|^2 &\le \left[\alpha+\beta\frac{\left\{2\|x_{2n}-T_2y_{2n}\|^2+2\|T_2y_{2n}-T_1x_{2n}\|^2\right\}}{\left\{1+\left(2\|T_1x_{2n}-T_2y_{2n}\|^2+2\|T_2y_{2n}-x_{2n}\|^2\right)\right\}}\right]\left\{2\|T_1x_{2n}-T_2y_{2n}\|^2+2\|T_2y_{2n}-x_{2n}\|^2\right\}\\ &\le 2\alpha\|T_1x_{2n}-T_2y_{2n}\|^2+2\alpha\|T_2y_{2n}-x_{2n}\|^2+2\beta\|x_{2n}-T_2y_{2n}\|^2+2\beta\|T_2y_{2n}-T_1x_{2n}\|^2\\ &\le 2(\alpha+\beta)\|T_1x_{2n}-T_2y_{2n}\|^2+2(\alpha+\beta)\|T_2y_{2n}-x_{2n}\|^2\end{aligned}$$

$$i.e.\ \|T_1x_{2n}-T_2y_{2n}\|^2 \le \frac{2(\alpha+\beta)}{1-2(\alpha+\beta)}\|x_{2n}-T_2y_{2n}\|^2$$

Taking limit as $n\to\infty$, we get $\|T_1x_{2n}-T_2y_{2n}\|^2 \to 0.$ It follows that

$$\|x_{2n}-T_1x_{2n}\|^2 \le 2\|x_{2n}-T_2y_{2n}\|^2+2\|T_2y_{2n}-T_1x_{2n}\|^2 \to 0 \qquad as\ \ n\to\infty.$$

and $\|u-T_1x_{2n}\|^2 \le 2\|u-x_{2n}\|^2+2\|x_{2n}-T_2y_{2n}\|^2 \to 0 \quad as\ \ n\to\infty.$

From (7)

$$\begin{aligned}\|T_1x_{2n}-T_2u\|^2 &\le \left\{\alpha+\beta\frac{\|x_{2n}-T_1x_{2n}\|^2}{\left(1+\|x_{2n}-u\|^2\right)}\right\}\|u-T_2u\|^2\\ &= \left\{\alpha+\beta\frac{\|x_{2n}-T_1x_{2n}\|^2}{\left(1+\|x_{2n}-u\|^2\right)}\right\}\|u-x_{2n}+x_{2n}-T_2u\|^2\\ &= \left\{\alpha+\beta\frac{\|x_{2n}-T_1x_{2n}\|^2}{\left(1+\|x_{2n}-u\|^2\right)}\right\}\left\{2\|u-x_{2n}\|^2+2\|x_{2n}-T_2u\|^2\right\}\\ &= \left\{\alpha+\beta\frac{\|x_{2n}-T_1x_{2n}\|^2}{\left(1+\|x_{2n}-u\|^2\right)}\right\}\left\{2\|u-x_{2n}\|^2+4\|x_{2n}-T_1x_{2n}\|^2+4\|T_1x_{2n}-T_2u\|^2\right\}\end{aligned}$$

Taking limit as $n\to\infty$, we get $(1-4\alpha)\|T_1x_{2n}-T_2u\|^2 \le 0$ It follows that $\|T_1x_{2n}-T_2u\|^2 \to 0$

Finally $\|u-T_2u\|^2 = \|u-T_1x_{2n}+T_1x_{2n}-T_2u\|^2$

$$\leq 2\|u-T_1x_{2n}\|^2+2\|T_1x_{2n}-T_2u\|^2 \to 0 \text{ as } n\to\infty$$

Hence $u=T_2u$.

Similarly we can prove that $u=T_1u$

Thus u is a unique common fixed point of T_1 and T_2.This completes the proof.
Letting $T_1 = T_2 = T$ in above theorem, we obtain the following

Corollary: Let X be a Hilbert space and C be a closed convex subset of X. Let T be a self mapping satisfying (7) .If there exists a point x_0 such that the I- scheme for T defined by

$$y_n = \beta_n Tx_{2n}+(1-\beta_n)x_n\,, \quad n\geq 0$$

$$x_{n+1} = (1-\alpha_n)x_n+\alpha_n Ty_n \,, \quad n\geq 0$$

converges to a point u , then u is fixed point of T.

In the Ishikawa Scheme $\{\alpha_n\}$, $\{\beta_n\}$ satisfies $0\leq\alpha_n\leq\beta_n\leq 1$ *for all n.*

$\lim\beta_n = 0$ as $n\to\infty$ and $\sum\alpha_n\beta_n = 0$.

Assuming that

(i) $0\leq\alpha_n,\beta_n\leq 1$, for all n.

(ii) $\lim\limits_{n\to\infty}\alpha_n=\alpha>0$,

(iii) $\lim\limits_{n\to\infty}\beta_n=\beta>1$

Theorem 2: Let X be a Hilbert space and let C be a closed convex subset or X. Let T_1 and T_2 be two mappings satisfying for some positive integers s and t :

$$\|T_1{}^s x-T_2{}^t y\| \leq \left\{\alpha+\beta\frac{\|x-T_1{}^s x\|^2}{\left(1+\|x-y\|^2\right)}\right\}\|y-T_2{}^t y\|^2 \qquad \ldots(11)$$

Where $\alpha+\beta<1/4$.

If there exists a point x_0 such that the I- scheme for T_1 and T_2 defined by (2) and (3), converges to a point u, $x_{2n}\to u,\ \|x_{2n+1}-x_{2n}\|\to 0.$ then u is a common point of T_1 and T_2 .

Proof: It follows from (2) that $x_{2n+1}-x_{2n} = \alpha_{2n}(T_2^t\, y_{2n}-x_{2n})$.

Since $x_{2n}\to u,\ \|x_{2n+1}-x_{2n}\|\to 0.$

Since $\{\alpha_{2n}\}$ is bounded away from zero, $\|T_2{}^t y_{2n}-x_{2n}\|\to 0$ as $n\to\infty$.

It follows that $\|u-T_2{}^t y_{2n}\|\to 0$ as $n\to\infty$.

Since T_1 and T_2 satisfy (7) we have,

$$\|T_1^s x_{2n}-T_2{}^t y_{2n}\|^2 \leq \left\{\alpha+\beta\frac{\|x_{2n}-T_1^s x_{2n}\|^2}{\left(1+\|x_{2n}-y_{2n}\|^2\right)}\right\}\|y_{2n}-T_2{}^t y_{2n}\|^2 \qquad \ldots(12)$$

Now $\|y_{2n}-x_{2n}\|^2=\|\beta_{2n}T_1^s x_{2n}+(1-\beta_{2n})x_{2n}-x_{2n}\|^2$

$$=\|\beta_{2n}T_1^s x_{2n}+x_{2n}-\beta_{2n}x_{2n}-x_{2n}\|^2$$

$$=\|\beta_{2n}(T_1^s x_{2n}-x_{2n})\|^2$$

$$=\beta_{2n}{}^2\|(T_1^s x_{2n}-T_2^t y_{2n})+(T_2^t y_{2n}-x_{2n})\|^2$$

$$\le 2\|T_1^s x_{2n}-T_2^t y_{2n}\|^2+2\|(T_2^t y_{2n}-x_{2n})\|^2 \quad ...(13)$$

and $\|y_{2n}-T_2^t y_{2n}\|^2=\|\beta_{2n}T_1^s x_{2n}+(1-\beta_{2n})x_{2n}-T_2^t y_{2n}\|^2$

$$=\|\beta_{2n}T_1^s x_{2n}+(1-\beta_{2n})x_{2n}-T_2^t y_{2n}+\beta_{2n}T_2^t y_{2n}-\beta y_{2n}T_2^t y_{2n}\|^2$$

$$=\|\beta_{2n}(T_1^s x_{2n}-T_2^t y_{2n})+(1-\beta_{2n})(x_{2n}-T_2^t y_{2n})\|^2$$

$$\le 2\beta^2{}_{2n}\|T_1^s x_{2n}-T_2^t y_{2n}\|^2+2(1-\beta_{2n})^2\|x_{2n}-T_2^t y_{2n}\|^2$$

$$\le 2\|T_1^s x_{2n}-T_2^t y_{2n}\|^2+2\|x_{2n}-T_2^t y_{2n}\|^2 \quad ...(14)$$

From (13) and (14),(12) can be written as:

$$\|T_1^s x_{2n}-T_2^t y_{2n}\|^2\le\left[\alpha+\beta\frac{\left\{2\|x_{2n}-T_2^t y_{2n}\|^2+2\|T_2^t y_{2n}-T_1^s x_{2n}\|^2\right\}}{\left\{1+\left(2\|T_1^s x_{2n}-T_2^t y_{2n}\|^2+2\|T_2^t y_{2n}-x_{2n}\|^2\right)\right\}}\right]\left\{2\|T_1^s x_{2n}-T_2^t y_{2n}\|^2+2\|T_2^t y_{2n}-x_{2n}\|^2\right\}$$

$$\le 2\alpha\|T_1^s x_{2n}-T_2^t y_{2n}\|^2+2\alpha\|T_2^t y_{2n}-x_{2n}\|^2+2\beta\|x_{2n}-T_2^t y_{2n}\|^2+2\beta\|T_2^t y_{2n}-T_1^s x_{2n}\|^2$$

$$\le 2(\alpha+\beta)\|T_1^s x_{2n}-T_2^t y_{2n}\|^2+2(\alpha+\beta)\|T_2^t y_{2n}-x_{2n}\|^2$$

i.e. $\|T_1^s x_{2n}-T_2^t y_{2n}\|^2\le\dfrac{2(\alpha+\beta)}{1-2(\alpha+\beta)}\|x_{2n}-T_2^t y_{2n}\|^2$

Taking limit as $n\to\infty$, we get $\|T_1^s x_{2n}-T_2^t y_{2n}\|^2\to 0.$ It follows that

$$\|x_{2n}-T_1^s x_{2n}\|^2\le 2\|x_{2n}-T_2^t y_{2n}\|^2+2\|T_2^t y_{2n}-T_1^s x_{2n}\|^2\to 0 \quad as\ n\to\infty.$$

and $\|u-T_1^s x_{2n}\|^2\le 2\|u-x_{2n}\|^2+2\|x_{2n}-T_2^t y_{2n}\|^2\to 0\quad as\ n\to\infty.$

From (11)

$$\|T_1^s x_{2n}-T_2^t u\|^2\le\left\{\alpha+\beta\frac{\|x_{2n}-T_1^s x_{2n}\|^2}{\left(1+\|x_{2n}-u\|^2\right)}\right\}\|u-T_2^t u\|^2$$

$$=\left\{\alpha+\beta\frac{\|x_{2n}-T_1^s x_{2n}\|^2}{\left(1+\|x_{2n}-u\|^2\right)}\right\}\|u-x_{2n}+x_{2n}-T_2^t u\|^2$$

$$=\left\{\alpha+\beta\frac{\|x_{2n}-T_1^s x_{2n}\|^2}{\left(1+\|x_{2n}-u\|^2\right)}\right\}\left\{2\|u-x_{2n}\|^2+2\|x_{2n}-T_2^t u\|^2\right\}$$

$$=\left\{\alpha+\beta\frac{\|x_{2n}-T_1^s x_{2n}\|^2}{\left(1+\|x_{2n}-u\|^2\right)}\right\}\left\{2\|u-x_{2n}\|^2+4\|x_{2n}-T_1^s x_{2n}\|^2+4\|T_1^s x_{2n}-T_2^t u\|^2\right\}$$

Taking limit as n $\to\infty$, we get $(1-4\alpha)\|T_1^s x_{2n}-T_2^t u\|^2\le 0$ It follows that $\|T_1^s x_{2n}-T_2^t u\|^2\to 0$

Finally $\|u-T_2^t u\|^2 = \|u-T_1^s x_{2n}+T_1^s x_{2n}-T_2^t u\|^2$

$$\le 2\|u-T_1^s x_{2n}\|^2+2\|T_1^s x_{2n}-T_2^t u\|^2\to 0 \; as \;\; n\to\infty$$

Hence $u=T_2^t u$.

Similarly we can prove that $u=T_1^s u$

Thus u is a unique common fixed point of T_1^s and T_2^t. Hence u is common fixed point of T_1 and T_2.

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