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Some More Remarks on Generalized Useful Information Measure

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ABSTRACT: Let

$$ {}_tL_u = \frac{1}{t}\log_D\left(\sum_{i=1}^{k}\frac{p_i^{\beta}u_i^{t+1}}{\left(\sum_{j=1}^{k}p_j^{\beta}u_j\right)^{t+1}}D^{n_i t}\right) \quad (t\neq 0) \qquad \ldots(1)$$

$$ {}_{\alpha}H(P,U) = \frac{1}{1-\alpha}\log\sum_{i=1}^{k}\frac{u_i p_i^{\alpha+\beta-1}}{\sum u_j p_j^{\beta}} \qquad \ldots(2)$$

Where p_i is the probability of the *i* th input symbol to a noiseless channel, n_i is the length of the code sequence for the *i* th symbol in some uniquely decipherable code and u_i is the utility factor. This utility factor has very important significance in communication problems. We shall find its lower and upper bounds in terms of generalized useful information.

In this paper we shall find a relation between quantities (1) and (2) using the relation $\sum D^{n_i}p_i^{\beta-1} \le 1$.

I. INTRODUCTION

Consider the following model for a finite random experiment

$$ S = \begin{bmatrix} a_1 & a_2 & \cdots & a_k \\ p_1 & p_2 & \cdots & p_k \\ u_1 & u_2 & \cdots & u_k \end{bmatrix} \qquad \ldots(3)$$

Where A=$(a_1, a_2, \ldots\ldots.., a_k)$ is the alphabet,

P = $(p_1, p_2, \ldots\ldots.., p_k)$ is the probability distribution and

U = $(u_1, u_2, \ldots\ldots.., u_k)$ is the utility distribution. The u_i are non-negative real numbers.

Consider

$$ H(P,U) = -\sum_{i=1}^{k}p_i^{\beta}u_i\log p_i \qquad \ldots(4)$$

Consider the problem of encoding the letters output by S in (3) by means of a single letter prefix code, whose code-words $(w, w, \ldots\ldots.., w_k)$ have lengths $(n_1, n_2, \ldots\ldots.., n_k)$ satisfying the inequality

$$ \sum_{i=1}^{k}D^{n_i}p_i^{\beta-1} \le 1 \qquad \ldots(5)$$

Here D is the size of code alphabet.

The useful mean length L_u of the code was defined as:

$$ L_u = \frac{\sum u_i n_i p_i}{\sum u_i p_i} \qquad \ldots(6)$$

And the authors obtained bounds for it in terms of (P, U) .

In this paper, we study coding theorems by considering a new function depending on the parameters α and β and a utility function. Our motivation for studying this new function is that it generalizes “useful” information measure.

Consider the function

$$ {}_tL_u = \frac{1}{t}\log_D\left(\sum_{i=1}^{k}\frac{p_i^{\beta}u_i^{t+1}}{\left(\sum_{j=1}^{k}p_j^{\beta}u_j\right)^{t+1}}D^{n_i t}\right) \quad (t\neq 0) \qquad \ldots(7)$$

Which we call as the function exponential useful mean length of code-words weighted with the function of probabilities and utilities.

Consider also the function

$$ \alpha H(P,U) = \frac{1}{1-\alpha}\log\sum_{i=1}^{k}\frac{u_i p_i^{\alpha+\beta-1}}{\sum u_j p_j^{\beta}} \qquad \ldots(8)$$

We call this as satisfactory measure for the valuable or useful information.

In the next section we now find a relation between the quantities (7) and (8) under the condition $\sum_{i=1}^{k} D^{n_i} p_i^{\beta-1} \le 1$.

Theorem 1: For every uniquely decipherable code, the generalized α- average length of codewords satisfies

$$\frac{\alpha}{1-\alpha} log_D \left(\sum_{i=1}^{k} \frac{p_i^{\beta} u_i^{\frac{1}{\alpha}} D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)}}{\left(\sum_{j=1}^{k} p_j^{\beta} u_j\right)^{\frac{1}{\alpha}}} \right) \ge \frac{\frac{1}{1-\alpha} log_2 \left(\sum_{i=1}^{k} \frac{u_i p_i^{\alpha+\beta-1}}{\left(\sum_{j=1}^{k} u_j p_j^{\beta}\right)} \right)}{log_2 D} \quad \ldots(9)$$

Whenever $\alpha > 0, D \ge 2,$ n_i are integers, $p_i \ge 0 \ (i = 1,2, \ldots \ldots, k)$ and $\sum_{i=1}^{k} D^{n_i} p_i^{\beta-1} \le 1, \quad \sum_{i=1}^{k} p_i = 1.$

Proof: We shall use the Holder's inequality

$$\sum x_i y_i \ \ge \ \left(\sum x_i^p\right)^{\frac{1}{p}} \left(\sum y_i^q\right)^{\frac{1}{q}} \quad \ldots(10)$$

if $p < 1$ ($\neq 0$) and $p^{-1} + q^{-1} = 1$.

There is equality in (12) if and only if there exist a positive number c such that

$$x_i^p = c y_i^q$$

Let us take

$$x_i = p_i^{-\frac{\beta}{t}} \left(\frac{u_i}{\sum_{j=1}^{k} p_j^{\beta} u_j} \right)^{-\frac{t+1}{t}} D^{-n_i} \quad \ldots(11)$$

$$y_i = p_i^{\frac{\alpha+\beta-1}{\alpha t}} \left(\frac{u_i}{\sum_{j=1}^{k} p_j^{\beta} u_j} \right)^{\frac{t+1}{t}} \quad \ldots(12)$$

Since $p = -t$ and so $q = \frac{p}{1-p} = \frac{t}{t+1}$

We put values from equations (11), (12) in (10) and using the generalized Craft's inequality

$\sum_{i=1}^{k} D^{n_i} p_i^{\beta-1} \le 1$, we have

$$1 \ge \sum_{i=1}^{k} D^{n_i} p_i^{\beta-1} \ge$$

$$\ge \left[\sum_{i=1}^{k} \left(p_i^{-\frac{\beta}{t}} \left(\frac{u_i}{\sum_{j=1}^{k} p_j^{\beta} u_j} \right)^{-\frac{t+1}{t}} D^{-n_i} \right)^{-t} \right]^{-\frac{1}{t}} \times$$

$$\left[\sum_{i=1}^{k} \left(p_i^{\frac{\alpha+\beta-1}{\alpha t}} \left(\frac{u_i}{\sum_{j=1}^{k} p_j^{\beta} u_j} \right)^{\frac{t+1}{t}} \right)^{\frac{t}{1+t}} \right]^{\frac{t+1}{t}}$$

Or

$$\left[\sum_{i=1}^{k} p_i^{\beta} \left(\frac{u_i}{\sum_{j=1}^{k} p_j^{\beta} u_j} \right)^{t+1} D^{n_i t} \right]^{\frac{1}{t}} \ge \left[\sum_{i=1}^{k} p_i^{\frac{\alpha+\beta-1}{\alpha(t+1)}} \left(\frac{u_i}{\sum_{j=1}^{k} p_j^{\beta} u_j} \right) \right]^{\frac{t+1}{t}}$$

Taking log on both the sides, we have

$$\frac{1}{t} log_D \left[\sum_{i=1}^{k} p_i^{\beta} \left(\frac{u_i}{\sum_{j=1}^{k} p_j^{\beta} u_j} \right)^{t+1} D^{n_i t} \right] \ge \frac{t+1}{t} log_D \left[\sum_{i=1}^{k} p_i^{\frac{\alpha+\beta-1}{\alpha(t+1)}} \left(\frac{u_i}{\sum_{j=1}^{k} p_j^{\beta} u_j} \right) \right]$$

Put $\alpha = \frac{1}{t+1}, \alpha > 0, \alpha \neq 1$ we have

$$\frac{\alpha}{1-\alpha} log_D \left(\sum_{i=1}^{k} \frac{p_i^{\beta} u_i^{\frac{1}{\alpha}} D^{n_i\left(\frac{1-\alpha}{\alpha}\right)}}{\left(\sum_{j=1}^{k} p_j^{\beta} u_j\right)^{\frac{1}{\alpha}}} \right) \ge \frac{\frac{1}{1-\alpha} log_2 \left(\sum_{i=1}^{k} \frac{u_i p_i^{\alpha+\beta-1}}{\left(\sum_{j=1}^{k} u_j p_j^{\beta}\right)} \right)}{log_2 D}$$

This proves the theorem.

Particular case: If $\beta = 1$ *and* $\alpha \to 1$ then (9) reduces to the result obtained by G. Longo [4].

$$L_u \ge \frac{H(P,U) - \overline{ulogu} + \overline{u} log \overline{u}}{\overline{u} log D}$$

Theorem 2: By properly choosing the lengths $n_1, n_2, \ldots \ldots \ldots, n_k$ in the code of theorem 1, $_tL_u$ can be made to satisfy the following inequality:

$$\frac{\alpha}{1-\alpha} log_D \left(\sum_{i=1}^{k} \frac{p_i^{\beta} u_i^{\frac{1}{\alpha}} D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)}}{\left(\sum_{j=1}^{k} p_j^{\beta} u_j\right)^{\frac{1}{\alpha}}} \right) < \frac{\frac{1}{1-\alpha} log_2 \left(\sum_{i=1}^{k} \frac{u_i p_i^{\alpha+\beta-1}}{\left(\sum_{j=1}^{k} u_j p_j^{\beta}\right)} \right)}{log_2 D} + 1 \quad \ldots(13)$$

Proof: Let n_i be the (unique) positive integer satisfying the inequality

$$-log_D \left(\frac{u_i p_i^{\alpha}}{\sum_{j=1}^{k} u_j p_j^{\alpha+\beta-1}} \right) \le n_i < -log_D \left(\frac{u_i p_i^{\alpha}}{\sum_{j=1}^{k} u_j p_j^{\alpha+\beta-1}} \right) + 1 \quad i = 1,2, \ldots \ldots, k \quad \ldots(14)$$

From the left inequality of (14)

$$D^{-n_i} \le \left(\frac{u_i p_i^{\alpha}}{\sum_{j=1}^{k} u_j p_j^{\alpha+\beta-1}} \right) \qquad i = 1,2, \ldots \ldots, k$$

Multiplying by $p_i^{\beta-1}$ and summing, we get:

$$\sum_{i=1}^{k} D^{n_i} p_i^{\beta-1} \leq 1 \qquad ...(15)$$

This is generalized Craft's inequality. Therefore there indeed exist uniquely decipherable codes with the code word length determines by (15).

From the right inequality of (15)

$$n_i < -log_D\left(\frac{u_i p_i^{\alpha}}{\sum_{j=1}^{k} u_j p_j^{\alpha+\beta-1}}\right) + 1$$

Or

$$D^{-n_i} > \left(\frac{u_i p_i^{\alpha}}{\sum_{j=1}^{k} u_j p_j^{\alpha+\beta-1}}\right) D^{-1}$$

Or

$$D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)} > \left(\frac{u_i p_i^{\alpha}}{\sum_{j=1}^{k} u_j p_j^{\alpha+\beta-1}}\right)^{\left(\frac{\alpha-1}{\alpha}\right)} D^{\left(\frac{1-\alpha}{\alpha}\right)}$$

Multiplying both sides by $p_i^{\beta}\left(\frac{u_i}{\sum_{j=1}^{k} p_j^{\beta} u_j}\right)^{\frac{1}{\alpha}}$ and summing and then taking log on both sides, we obtain result.

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