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Spectral Measures and Spectral Families

Nitesh Kumar* and D.C. Gokhroo**
[*]Associate Professor, Department of Mathematics,
Dronacharya College of Engineering, Gurgaon, (HR)
[**]Principal, Rajasthan Higher Education Services, Ajmer, (RJ)

(Corresponding author Nitesh Kumar)
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ABSTRACT: We focus here on some very recent results and its studies. Our main contribution is to provide some numerical and empirical facts concerning spectral Measures and Spectral Families.

I. INTRODUCTION

E. Kowalski [1] follows up on a standard course in Functional Analysis and builds on the principles of functional analysis to discuss one of the must useful and widespread among its applications, the analysis, through spectral theory, of linear operators T: H1 → H2 between Hilbert spaces. Roland Strömberg [2] discussed the the proofs of the book "Functional analysis" given by Reed and Simon. Antoine jacquier studied numerical approach to spectral risk measures.

Proposition 1: *Let H be a complex Hilbert space, and E a spectral family on H. The following two conditions are equivalent. (i)* $K_1 = var_2(E) < \infty$ *and* $K_2 = var_2(E^*) < \infty$*;*
(ii) For each $x \in H$ *,* $u \in P_R$*, the series* $\sum_{k \in Z} \Delta_k x$ *converges unconditionally, uniformly in u.*

Lemma 1: Let $2 \leq p < \infty$ and suppose that $K = var_p(E^*) < \infty$. Then

$$\|x\| \leq K\left(\sum_{\infty}^{\infty} \|\Delta_k x\|^{p'}\right)^{1/p'} \text{ for each } x \in H\ u \in P_R.$$

Proof. Let a > 0 and $x \in \{E(a)-E(-a)\}$ H and write $\xi = \frac{x}{\|x\|}$ so that$(x, \xi) = \|x\|$. Let $u \in P_R$. Then there exists some $N \geq 1$ such that $x = \sum_{-N}^{N} \Delta_k x.$ Hence we have

$$\|x\| = (\zeta, \sum_{|k|<N} \Delta_k x) = \sum_{|K|<N} (\zeta, \Delta_k x)$$

$$\sum_{|K|<N} (\zeta, \Delta^2{}_k x) = \sum_{|K|<N} (\Delta^*{}_k\, \zeta, \Delta_k x)$$

So, using Holder's inequality in the last term we have

$$\|x\| \leq \left(\sum_{|k|<N} \|\Delta_k x\|^{p'}\right)^{1/p'} \sum_{\infty}^{\infty} \|\Delta_k x * \zeta\|^{p'})^{1/p}$$

Hence

$$\|x\| \leq K\left(\sum_{-\infty}^{\infty} \|\Delta_k x\|^{p'}\right)^{1/p'} \qquad \ldots(3.1)$$

Since $U_{a>1}\{E(a)-E(-a)\}$ H is dense in H, the last inequality holds for all $x \in H$. For suppose not; then there exists $x \in H$ and $\varepsilon > 0$ such that $\| x - \sum_{-M}^{M} \Delta_k x \| < \frac{\in}{2}$,

Then we have

$$\left(\sum_{-\infty}^{\infty} \|\Delta_k x\|^{p'}\right)^{1/p'} + \left(\frac{\in}{2} = \|x\| - \frac{\in}{2} < \|x\| - \|x - \sum_{-M}^{M} \Delta_k x\| \leq \|\sum_{-M}^{M} \Delta_k x\|\right.$$

But $\|\sum_{-M}^{M}\Delta_k x\| \le K(\sum_{-M}^{M}\Delta_k x\|^{p'})^{1/p'}$ Hence we get $(\sum_{-\infty}^{\infty}\|\Delta_k x\|^{p'})^{1/p'} + \frac{\in}{2K} < (\sum_{-\infty}^{\infty}\|\Delta_k x\|^{p'})^{1/p'}$ which is a contradiction.

Proof of Proposition 3.1

Let us show (i)→(ii). Let $K_1 = var_2$ (E) and $K_2 = var_2$ (E*). Using Lemma 3.1 with p=2 we have, for each x ∈ H and u ∈ P_R,

$$\frac{1}{K_1} < \sum_{-\infty}^{\infty}\|\Delta_k x\|2)^{1/2} \le \|x\| \le K_2(\sum_{-\infty}^{\infty}\|\Delta_k x\|^2)^{1/2}$$

This shows that for each x ∈ H and $\{\lambda_k\}$ ∈ P_R, $\sum_{-\infty}^{\infty}\Delta_k x$ converges unconditionally. To see this, let ε∈ D^∞ and □>0.

Since $\{E(\lambda_N)\text{-}E(\lambda_{-N})\}x \to x$ as N→∞, we can find $N_0 \ge 1$ such that

$$(\sum_{N<|k|<M}\|\Delta_k x\|^2)^{1/2} \le \frac{\delta}{K_2} \text{ for all } N_0 \le N < M. \qquad \ldots(3.2)$$

Then we have

$$\|\sum_{|k|\le N}\in_k \Delta_k x\| \le K_2(\sum_{k=-\infty}^{\infty}\|\Delta_k x\|^2)^{1/2} \le \square \text{ for all } N_0 \le N < M. \qquad \ldots(3.3)$$

Hence $\sum_{-\infty}^{\infty} \in k\Delta_k x$ is Cauchy and so converges to some $I_{u,\varepsilon}$∈ H. Furthermore, (3.3) shows that the convergence is uniform in *u*, in the sense defined in the above Remark.

To show (ii) □ (i), suppose that, for each *x* □ *H*, the series $\sum \Delta_{kx}$ converges unconditionally, uniformly in *u*. So there exists a constant $M_X > 0$ such that

$$\|\sum_{k=-\infty}^{\infty}\in_k \Delta_k x\| \le M_x \text{ for all } u\in P_R, \varepsilon\in D^\infty.$$

But this means that all the balanced partial sums are also bounded:

$$\|\sum_{|k|\le k}\in_k \Delta_k x\| \le M_x \text{ for all } K\ge 1, u\in P_R, \varepsilon\in D^\infty.$$

To see this, let us fix $\{E_k\}$. Define $\in'_k = \in''_k = \in_k$ for $|k| \le K$ and $\in'_k - \in''_k = 1$ for $|k| > K$. Then we have

$$2\|\sum_{|k|\le k}\in_k \Delta_k x\| = \|\sum_{k=-\infty}^{\infty}\in'_k\Delta_k x\|$$

$$\|\sum_{k=-\infty}^{\infty}\in'_k\Delta_k x\| + \|\sum_{k=-\infty}^{\infty}\in''_k\Delta_k x\| \le 2M_x.$$

Hence we can apply the Uniform Boundedness Principle to the collection of operators $\{\sum_{|k|\le k}\in_k \Delta_k x : K\ge 1, u\in P_R \in\in D^\infty\}$ to deduce the existence of a constant C>0 such that $\|\sum_{k=-\infty}|k|\le K\in_k\Delta_k\| \le C$ for all $K\ge 1$, u ∈ P_R and $\in\in D^\infty$. Hence $\|\sum_{-\infty}^{\infty}\in_k \Delta_k x\| \le C\|x\|$ for all x ∈ H.

So the operators $A_{e,u} = \sum_{-\infty}^{\infty} \in_k \Delta_k$ are uniformly bounded by C, i.e. the collection G = {$A_{e,u}$: u $\in P_R \in \in D^{\infty}$} is bounded above by C. Observe that this is an fact a well-defined Abelian group. For, let $\Delta^{(u)}$ and $\Delta^{(v)}$ correspond to parturitions u and v $\in P_R$ respectively, and let $A_{e,u} = \sum_{-\infty}^{\infty} \in_k \Delta_k^{(u)}$ and $A_{\delta v} \equiv \sum_{-\infty}^{\infty} \delta_k \Delta_k^{(v)}$. Let we be the union of the points of u and v (so that if refines both). Now rewrite $A_{e,u} = \sum_{-\infty}^{\infty} \bar{\in}_j \Delta_k^{(w)}$, where for each j, we define $\in_j = \in_k$, with k being the unique integer for which $\Delta_j(w)\Delta_k(u) \neq 0$. We can rewrite $A_{\delta,u} = \sum_{-\infty}^{\infty} \tilde{\delta} \Delta_j^{(w)}$, in terms of w in an exactly analogous manner. Then the product of the two operators is uniquely given by:

$$A_{\in u} A_{\delta,v} = \sum_{-\infty}^{\infty} \in_j \delta_j \Delta_j^{(w)}$$

We now apply a result of B. Sz. Nagy: there exists an inner product (,) on *H* , equivalent to the original (,), with respect to which all $A_{\epsilon,u}$ are unitary. That is, there exist constants C_1 and C_2 such that

$$(A_{\epsilon,u}x, A_{\epsilon,u}y) \quad = \quad (x, y) \text{ for all } x, y \,\square\, H$$

$$C_1(x, x) \le (x, x) \le C_2(x, x) \text{ for all } x \,\square\, H.$$

Now, for any partition *u*, the operators {Δ_k } are orthogonal with respect to this new inner product, in the sense that ($\Delta_k x, \Delta_j x$) = 0 if $k \neq j$. To see this, let us fix a partition *u* and $k \neq j$. Choose □ □ D^{∞} such that □$_j$ □$_k$ = −1. Then, using the unitary property, we have ($A_{\square,u}\Delta_k x, A_{\square,u}\Delta_j x$) = ($\Delta_k x, \Delta_j x$). But we also have $A_{\square,u}\Delta_i x$ = □$_i \Delta_i x$ for any *i* □ Z so that

($A_{\square,u}\Delta_k x, A_{\square,u}\Delta_j x$) = □$_k$ □$_j$ ($\Delta_k x, \Delta_j x$) = −($\Delta_k x, \Delta_j x$).

Combining the last two equations we get ($\Delta_k x, \Delta_j x$) = −($\Delta_k x, \Delta_j x$) = 0, as claimed.

Now let *a* > 0 and *x* □ $\{E(a) - E(-a)\}H$. So $x = \sum_{-M}^{M} \Delta_k x$ for some M≥1.

Then denoting the new norm by $\|x\|^2_N \equiv \{x, x\}$,

$$(x, x) = (\sum_{-M}^{M} \Delta_k x, \sum_{-M}^{M} \Delta_j x) = \sum_{-M}^{M} \Delta_k x, \|^2_N)^{1/2}$$

Hence we have

$$C_1 \sum_{-M}^{M} \| \Delta_k x \|^2_N) \le \| x \|^2 \le C_2 \sum_{-M}^{M} \| \Delta_k x \|^2_N \qquad \ldots(3.4)$$

But we also have $\frac{1}{C_2} \| \Delta_k x \|^2_N \le \Delta_k x \|^2_N \le \frac{1}{C_1} \| \Delta_k x \|^2)^{1/2}$

This holds for any *x* □ $\{E(a) - E(-a)\}H$, and any partition *u*. But Ua>0{E(*a*) − $E(-a)\}H$ is dense in *H* , so (†) holds for all *x* □ *H* . So, taking the supremum over u $\in$PR and x $\in B_H$ in the left inequality, we deduce that var2(E) < $\sqrt{\frac{C_2}{C_1}}$.

Let us show the var2 (E*)≤2 $\|E\|\infty \sqrt{\frac{C_2}{C_1}}$. Again, let ξ $\in$ {E*(a)-E*(-a)}H for some fixed a>0 and let u $\in P_R$. So there exists some N≥1 such that $\xi = \sum_{-N}^{N} \Delta * k\xi$. Set $z_i = \Delta * i\zeta$ for I =-N,………N. Then we have for each i

$$\| \Delta_i z_i \| = \Delta_i \Delta_i * \zeta \| \le 2 \| \infty \| \Delta_i * \zeta \|$$

Now set z = $\sum_{-N}^{N} \Delta_i z_i$.

Then

$\Delta k^{z} = \Delta k^{z} k$ for k = -N,..,N, …(3.5)

$(z, \Delta^{*}{}_{k}\zeta) = (\Delta k z_k, \zeta) = (z_k, \Delta * k\zeta) = \| \Delta^{*} k\zeta \|^2$

Now by () we have $\|z\| \le \sqrt{\frac{C_2}{C_1}(\sum_{-N}^{N} \| \Delta k z \|^2)^{1/2}}$ and also $|(z,\zeta)\| \le \| z \| \; \|\zeta\|$, so that

$$\|\xi\| \ge \frac{|(z,\zeta)|}{\|z\|} \ge \frac{|(z,\zeta)|}{\sqrt{\frac{C_2}{C_1}(\sum_{-N}^{N} \| \Delta_k z \|^2)^{1/2}}}$$

But now we have from (+) and (3.5)

$$\sum_{|k|\le N} \| \Delta_k z |^2)^{1/2} \le 2 \| E \| \infty (\sum_{|k|\le N} \| \Delta * k\zeta \|^2)^{1/2}$$

$$(z,\zeta) = \sum_{|k|\le} \| \Delta^{*}{}_{k}\zeta \|^2$$

Hence,

$$\| \zeta \| \ge \frac{\sum_{-N}^{N} \| \Delta^{*}{}_{k}\zeta \|^2}{2 \| E \| \infty \sqrt{\frac{C_2}{C_1} \sum_{-N}^{N} \| \Delta^{*}{}_{k}\zeta \|^2)^{1/2}}} = \frac{1}{2 \| E \| \infty \sqrt{\frac{C_2}{C_1}}} (\sum_{-N}^{N} \| \Delta^{*}{}_{k}\zeta \|^2)^{1/2} \quad \ldots(3.6)$$

So we have

$$\sum_{-N}^{N} \| \Delta^{*}{}_{k}\zeta \|^2)^{1/2} \le 2 \| E \| \infty \sqrt{\frac{C_2}{C_1}} \| \zeta \|$$

This holds for any $\xi \in U_{a>0}$ {E*(a)-E*(-a)}H and any partition u $\in P_R$, so taking supreme we obtain.

$\operatorname{var} 2(E^*) \le 2 \| E \| \infty \sqrt{\frac{C_2}{C1}}$ as required

This Proposition helps establish the main result of this chapter.

Theorem 1: *Let E be a spectral family on a complex Hilbert space H . If both* $var_2(E) = K_1 < \infty$ *and* $var_2(E^*) = K_2 < \infty$, *then E gives rise to a spectral measure on* B, *the Borel σ−algebra on* R. *That is there exists a spectral measure* E *on* B *such that for any A* = (*a*, *b*] □ R, $\varepsilon(A) = \{E(b) - E(a)\}$.

Proof. Let us use the notation from the statement and proof of Proposition 3.1. We have shown therein that provided $var_2(E) < \infty$ and $var_2(E^{□}) < \infty$, the operators $Ae,u \equiv \sum_{-\infty}^{\infty} \in_k \Delta_k$ are well defined and bounded and moreover the Abelian group

$G \equiv \{A_{eu} : u \in P_R \in D^{\infty})$ is uniformly well bounded.

Now, by XV 6.1 in [3.7], there exists an invertible self-adjoint *S* □ B(*H*) such that for every $A_{e,u}$ □ *G*, the operator $B_{E,u} \equiv S^{-1} A_{e,u} S$ is unitary. Observe that, since $A^2_{e,u}$=1H, we have $B^2_{e,u}$ =1H = $B_{e,u}$ $B^{*} \in, u$, so that each $B \in u$ is self-adjoint.

Now, observe that for any *μ* □ R, *E*(*μ*) □ *G*. To see this, simply define *u* □ P_R to be Z, with the exception $\lambda_0 = \mu$. Then choose *E* □ D^{∞} to be $e_j = -1$ for $j \le 0$ and $e_j = 1$ for $j > 0$. Then we have $A_{e,u} = 1 - 2E(\mu)$ and hence

$$E(\mu) = \frac{1}{2}(I - A_{e,u}) \quad \ldots(3.7)$$

Now let $F(\mu) = S^{-1}E(\mu)S$. This is a well-defined spectral family in H and (3.7) gives $F(\mu) = \frac{1}{2}(I - B_{E,u})$ so that F is in fact self-adjoint. Let us now write $H_n = \{F(n) - F(n-1)\}H$ and $x_n = \{F(n) - F(n-1)\}x$ for $x \in H$.

Then H is a direct-sum decomposition. $H = \bigoplus_{-\infty}^{\infty} H_n$. Let

$$T_n = \int_R \lambda dF(d\lambda)H_n$$

Then T_n is a *bounded* self-adjoint operator on H_n. Hence by the classical Spectral

Theorem there exists a spectral measure F_n on the Borel σ-algebra B $(n-1, n]$ such that $T_n = \int_{n-1}^{n} \lambda F_n(d\lambda)$. Now define an operator valued set function.

$$F(A) \equiv \bigoplus_{-\infty}^{\infty} F_n.(A \cap (n-1, n]) \qquad A \in B(R)$$

First observe that F(A) is well-defined, since $F_n(A \cap (n-1, n)$ is a bounded operator from H_n into itself and so $F(A)x = \bigoplus_{-\infty}^{\infty} F_n.(A \cap (n-1), n])x_n$ is well-defined. In fact, F defines a projection-valued measure, for it satisfies the following three properties:

(i) $F(R) = I_H$;

(ii) if A, B $\in$ B(R) then F (A$\cap$ B)= F(A) F (B);

(iii) if $\{A_k\} \subset$ B(R) is a sequence of pairwise disjoint sets, then for each $x \in H$, $F(\cup_k A_k)x = \sum_{k=1}^{\infty} F(A_k)x$

(i) is trivially true, as $F_n((n-1, n]) = I_{H_n}$ for all n. (ii) is equally easy. For if $A, B \in$ B(R), then $A \cap (n-1, n]$, $B \cap (n-1, n] \subset$ B $(n-1, n]$ for each n. So, F_n being a spectral measure, we have

$$F_n(A \cap B \cap (n-1, n]) = F_n(A \cap (n-1, n])F_n(B \cap (n-1, n])$$

Hence

$$F(A \cap B) = \bigoplus_{-\infty}^{\infty} F_n(A \cap B \cap (n-1, n])$$

$$= \{\bigoplus_{-\infty}^{\infty} F_n(A \cap (n-1, n])\}\{\bigoplus_{-\infty}^{\infty} F_n(B \cap (n-1, n])\} \qquad \ldots(3.8)$$

= F(A) F (B)

(The equality in (3.8) is just the definition of the product of direct-sum operators.) Finally, to check (iii), let $\{A_k\} \subset$ B(R) be a sequence of disjoint Borel sets, set $A = \bigcup_{k=1}^{\infty} A_k$ and let $x \in H$. Using orthogonality of the spaces H_n we have

$$\langle F(A)x, x\rangle = \langle \bigoplus_{-\infty}^{\infty} F_n(A \cap (n-1, n])x, x\rangle = \sum_{n=-\infty}^{\infty} \langle F_n(A_k \cap (n-1, n])x_n, x_n\rangle \qquad \ldots(3.9)$$

Now, since each F_n is a spectral measure on $(n-1, n]$, we have

$$\langle F_n(A \cap (n-1, n])x_n, x_n\rangle = \sum_{k=1}^{\infty} \langle F_n(A_k \cap (n-1, n])x_n, x_n\rangle$$

Moreover, for every $n \in$ Z, $(F_n(\cdot)x_n, x_n)$ is a *positive* Borel measure. Hence we can swap the order of summation in line (3.10) below.

$$\langle F(A)x, x\rangle = \sum_{n=-\infty}^{\infty} \sum_{k=1}^{\infty} \langle F_n(A_k \cap (n-1, n])x_n, x_n\rangle$$

$$= \sum_{k=1}^{\infty} \{\sum_{n=-\infty}^{\infty} \langle F_n(A_k \cap (n-1, n])x_n, x_n\rangle\} \qquad \ldots(3.10)$$

$$= \sum_{k=1}^{\infty} \langle \underset{n=-\infty}{\overset{\infty}{\oplus}} \langle F_n(A_k \cap (n-1, n])x_n, x\rangle$$

$$= \sum_{k=1}^{\infty} \langle F(A_k x, x\rangle$$

To put this another way, $l\,\underset{n\to\infty}{im} \langle \{F(a) - \sum_{k=1}^{N} F(A_k)\}x, x\rangle = 0$. Now, the operation $F(A)$ and $\{\sum_{k=1}^{N} F(A)\} N \geq 1$ are self adjoint, so by polarization we have

$$\lim_{n\to\infty} \langle \{F(a) - \sum_{k=1}^{N} F(A_k)\}x, x\rangle = 0 \qquad \text{for all x, y} \in \text{H.}$$

Hence $\sum_{k=1}^{\infty} F(A_k)$ x converges weakly and so strongly to F(A) x, and this establishes (iii). Thus F is a genuine spectral measure.

Now, suppose, A = (a,b] is an interval such that n-1>a<n≤b<n+1 for some integer n ∈ Z. Then F_k (A∩ (k-1, k])=0 if k ≠n or n+1. Furthermore,

$$F_n(A \cap (n-1, n]) = Fn((a, n]) = \int_{n-1}^{n}{}_{(a,n)}(\lambda) F_n(d\lambda)$$

$$= \int_{n-1}^{n} \amalg_{(a,n)}(\lambda) dF(\lambda) H_n = \{F(n) - F(a)\} H_n$$

Similarly $F_{n+1}(A \cap (n, n+1)]) = \{F(b) - F(n)\} H_{n+1}$ and so writing somewhat clumsily,

$$F(A) = \underset{k=-\infty}{\overset{n-1}{\oplus}} 0H_k \oplus \{F(n) - F(a)\} H_n \oplus \{F(b) - F(n)\} H_{n+1} \oplus \underset{k=n+2}{\overset{\infty}{\oplus}} 0H_k$$

But this says precisely that F(A) = F(b) – F(a). In a similar manner we can show that F((c,d)]F(d)-F(c) for any interval (c,d]. So we finally define

ε(A) = SF(A)S-1 for A ∈ B(R)

ε(.) is then a well defined spectral measure on B(R) and the last calculation shows that satisfies ε(A) = {E (b)-E(a)} for a subset A = (a,b] ∈ R.

An Example of $var_s(E) = \infty$

Proposition 1 clearly shows that $var_2(E) < \infty$ and $var_2(E) < \infty$ is a very restrictive condition: it is equivalent to *E* being a spectral measure. It is of interest, therefore, to establish that not all spectral families on a Hilbert space exhibit this phenomenon. In fact, we can show more. Given any $s \geq 2$, there exists a Hilbert space *H* and a spectral family *E* on *H* such that $var_s(E) = \infty$. To achieve this, we shall construct a conditional basic sequence $\{e_k\}_{k\geq 1}$ in L^2 (T) and let $H = lin\{e_k\}$. Then we shall define a spectral family *E* and an x □ H , dependant on the given value of *s*, such that for all sufficiently fine partitions $u \in P_R, \sum_{-\infty}^{\infty} \| \Delta_k x \|^s{}_H = \infty$. The search for a suitable conditional basic sequence is motivated by [23], in particular the following theorem therein.

Theorem 2: Let 0<b<1/2 and 1<p<∞ satisfy $\frac{1}{p} > \frac{1}{2} - b. Let \{a_k\}_{k\geq 0}$ be a positive monotonic decreasing sequence such that $\sum_k^{\infty} a^p{}_k < \infty$. Then the series $\sum_{k=0}^{\infty} a_k(t)^b$ cos kt converges in L^2(T).

It is necessary for our basic sequence to be bounded below, and the following Lemma ensures that is the case.

Lemma 2 Let 0<b<$\frac{1}{2}$ and define functions $e_k\, L^2[-\pi, \pi]$ by $e_k(t) = |t|^b$ cos kt for k ≥0. Then there exists a constant M_b>0 such that $\|e_k\|_{L^2} \geq M_b$ for all k≥0.

Proof. For any k≥0 we have $\|e_k\|^2 = \int_{-\pi}^{\pi} |t|^{2b} \cos^2 kt\, dt = 2\int_0^{\pi} t^{2b} \cos^2 kt\, dt.$

Let us consider k≥3. Let

$$I_0 = \left[0, \frac{\pi}{3k}\right],$$

$$I_{j+1} = \left[\frac{(2+3j)\pi}{3k}, \frac{(4+3j)\pi}{3k}\right] \; j = 0,\ldots,k-2$$

Then $U_{0\le j\le k-2} I_j \subset [0,2\pi]$ and on each I_j we have $\cos^2 kt \ge \frac{1}{4}$. Hence

$$\| e_k \|^2 \ge 2\sum_{j=0}^{k-2}\int_{I_j} t^{2b}\cos^2 kt\,dt$$

$$\ge \frac{1}{2}\sum_{j=0}^{k-2}\int_{I_j} |t|^{2b}\,dt$$

$$\ge \frac{1}{2}\sum_{j=0}^{k-2}\frac{1}{1+2b}\left\{\left[\frac{4+3j}{3k}\pi\right]^{1+2b} - \left[\frac{2+3j}{3k}\pi\right]^{1+2b}\right\}$$

$$= \frac{1}{2(1+2b)}\left(\frac{\pi}{3k}\right)^{1+2b}\sum_{j=0}^{k-2}\left\{(4+3j)^{1+2b} - (2+3j)^{1+2b}\right\}$$

Now, the function $f(x) = (4+3x)^{1+2b}-(2+3x)^{1+2b}$ is increasing and concave on x≥0 so that

$$\int_0^{k-2} f(x)dx \le \sum_0^{k-2}\left\{(4+3j)^{1+2b} - (2+3j)^{1+2b}\right\}$$

Hence,

$$\frac{1}{2+2b}\left\{(3k-2)^{2+2b} - (3k-4)^{2+2b} - 4^{2+2b} + 2^{2+2b}\right\} \le \sum_0^{k-2}\left\{(4+3j)^{1+2b} - (2+3j)^{1+2b}\right\}$$

Substituting this into (3.11) we have for k≥3.

$$\| e_k \|^2 \ge \frac{\pi^{1+2b}}{2(1+2b)(2+2b)}\left\{\frac{(3k-2)^{2+2b} - (3k-4)^{2+2b} - 4^{2+2b} + 2^{2+2b}}{(3k)^{1+2b}}\right\}$$

Now the right hand side of this inequality is increasing as $k \to \infty$. Moreover, at $k = 3$ the right side is equal to

$$\frac{\pi^{1+2b}}{2(1+2b)(2+2b)}\left\{\frac{7^{2+2b} - 5^{2+2b} - 4^{2+2b} + 2^{2+2b}}{9^{1+2b}}\right\} > 0$$

Thus, for we have

$$\| e_k \|^2 \ge \frac{\pi^{1+2b}}{2(1+2b)(2+2b)}\left\{\frac{7^{2+2b} - 5^{2+2b} - 4^{2+2b} + 2^{2+2b}}{9^{1+2b}}\right\} \text{ for all } k \ge 3.$$

Further, we can let m = min $\{\|e_j^2\|$: j=0,1,2}>0 and then set

$$M^2{}_b = \min\{m, \frac{\pi^{1+2b}}{2(1+2b)(2+2b)}\left\{\frac{7^{2+2b} - 5^{2+2b} - 4^{2+2b} + 2^{2+2b}}{9^{1+2b}}\right\}\}$$

Proposition 2: *For any s ≥ 2 there exists a Hilbert space H and a spectral family* $E\{(\lambda)\}_{\lambda\square R}$ *on H such that* $var_S(E) = \infty$.

Proof. Let $P_R \cap [-a, a]$ denote the set of partitions of R, restricted to the interval $[-a, a]$. As, before, if $\{E(\lambda)\}$ is a spectral family and $u = (\mu_k)_{k \in Z} \in P_R$, then $\{\Delta_k\} = \{(E(\mu_k) - E(\mu_{k-1}))\}$ is the associated Schauder decomposition. Then by definition, var s(E) $\sup_{u\in P_R} \sup_{a>0} \sup_{u\in P_R \cap[-a,a]} \left(\sum_{k=-\infty}^{\infty} \| \Delta_k x^s \|\right)^{1/s} = \infty$

Therefore, given $2 \leq s < \infty$, it will suffice to construct a spectral family E and $x \in H$ such that

$$\sup_{u\in P_R} \left(\sum_{k=-\infty}^{\infty} \| \Delta_k x^s \|\right)^{1/s} = \infty$$

So, let s ≥ 2 be given. Choose $-\frac{1}{2} < a < 0$ and $s_1 > s$ such that

$$0 < \frac{1}{2} + a < \frac{1}{s_1} < \frac{1}{s} < \frac{1}{2}$$

Let $e_k(t) = |t|^{-a} \cos kt \in L^2[-\pi, \pi]$ for $k \geq 0$. By Lemma 2 there exists a constant $M_a > 0$ such that $\|e_k\|_{L^2} \geq M_a$ for all k. This is a conditional basic sequence in $L^2[-\pi, \pi]$ (see [1], so the space $H = lin\{e_k : k \geq 0\}$ is a Hilbert space.

Let $\{a_k\} \in l_{s_1}$ be given by $a_0 = 1$ and $a_k = \frac{1}{k^{1/s}}$ for $k \geq 1$. Now the basis $\{e_k\}$, the sequence $\{a_k\}$, and s_1 satisfy the conditions of Theorem 3.2, so that the sires $\sum_0^{\infty} a_k e_k$ converges in H. But we also note that $\left(\sum_0^{\infty} |a_k|^s\right)^{1/s} = \infty$.

Now we are ready to construct the required spectral family on H. Let $\{\lambda_k\}$ be a monotone strictly increasing sequence with $\lambda_0 = 0$ and $\lambda_k / 2\pi$. Let $\{\xi_k\}$ be the bi-orthogonal functionals associated with $\{e_k\}$ in the sense that $\langle e_k, \zeta_j \rangle = \int_{-\pi}^{\pi} e_k(t)dt = 0$ for $k \neq j$. Define.

$Pk : H \to H\ Y \to \langle y, \xi_k \rangle e_k$, for $k \geq 0$.

Now define E(μ) as follows

$$E(\mu) = 0 \ for \mu \in (-\infty, o),$$

$$E(\mu) = \sum_{j=0}^{k} P_j \ for \mu \in (\lambda_k, \lambda_{k+1}), \ k \geq 0.$$

$$E(\mu) = 1 \ for \mu \in (2\pi, \infty)$$

E is now a spectral family on H and is concentrated on $[0, 2\pi]$. Note that, in particular, $E(\lambda_k) - E(\lambda_{k-1}) = P_k$. Let $x = \sum_{k=-\infty}^{\infty} a_k e_k$ with $\{a_k\}$ as defined above. Since this sum converges in L^2 norm, x is a genuine element of H.

Claim

$$\sup_{u\in P_R} \left\{\sum_{k=-\infty}^{\infty} \| \Delta_k x \|^s \right\}^{1/s} = \infty$$

It suffices to show that for each $N \geq 1$ there exists a partition $u_N \in P_R$ such that $\left(\sum_{k=-\infty}^{\infty} \| \Delta_k x \|^s\right)^{1/s} \geq N$. So let $N \geq 1$ be given. Since $\left(\sum_{k=-\infty}^{\infty} | \Delta_k x |^s\right)^{1/s} = \infty$, we can pick J_N such that $\left(\sum_0^{J_N} | a_k |\right)^{1/s} \geq N$.

Let us define u_N as follows: let $\{\mu_k\}_{k\leq 0}$ be any partition of $(-\infty, 0]$ with $\mu_0 = 0$. Let $\mu_j = \lambda_j$ for $j = 1, ..., J_N$ and, without loss of generality, let $\mu_K = 2\pi$ for some $K > J_N$.
Finally, let $\{\mu_j\}_{j\geq K}$ be any partition of $[2\pi, \infty)$.

Thus, for any $y \in H$, we have $\Delta_k y = 0$ for $k < 0$ and $k > K$, and $\Delta_k y = P_k y$ for $0 \leq k \leq J_N$. Hence

$$\sum_{-\infty}^{\infty} \| \Delta_k x \|^s = \sum_{k=0}^{J_N} \| P_k y \|^s + \sum_{k=J_{N+1}}^{K} \| \Delta_k y \|^s \geq \sum_{k=0}^{J_N} \| P_k y \|^s$$

We now apply this to y=x and note that $P_k x = a_k e_k$ for $k \geq 0$. Thus we have

$$\left\{ \sum_{-\infty}^{\infty} \| \Delta_k x \|^s \right\}^{1/s} \geq \{ \sum_{k=0}^{J_N} | a_k | s \| e_k \|^s \}^{1/s}$$

$$\geq M_a \{ \sum_{k=0}^{J_N} | a_k |^s \}^{1/s} > N$$

This proves the Claim, and hence the Proposition

Thus we have settled the question of existence of a spectral family on a Hilbert space, which does not arise from a spectral measure. In fact the above construc tion gives a trigonometrically well bounded operator S_H = $\int_{0-}^{2\pi} e^{i\lambda} dE(\lambda)$ with interesting power growth properties.

II. CONCLUSION

In this paper we studied some theorems on Spectral Measures and Spectral Families

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