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A Common Fixed Point Theorem for Four Mappings in Hilbert Space using Rational Inequality

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ABSTRACT: The object of this paper is to obtain a common unique fixed-point theorem for four continuous mappings in Hilbert Space using rational inequality. Our result generalizes the results of P.V. Koparde and D.B. Wagmode.

Keywords: Hilbert Space, common fixed point.

Mathematics Subject Classification: Primary 47H10, Secondary 54H25.

I. INTRODUCTION

Kannan [3] proved that a self mapping T on complete metric space (X,d) satisfying the condition

$d(x,y) \leq \alpha\{ d(x,Tx)+d(y,Ty) \}$ for all x, y in X where $0 < \alpha < \frac{1}{2}$ has a unique fixed in (X,d).

Koparde and Waghmode [5] have proved fixed point theorem for a self mappings T on a closed subset S of Hilbert space H, satisfying the Kannan type condition

$$\|Tx-Ty\|^2 \leq \alpha\left\{\|x-Tx\|^2+\|y-Ty\|^2\right\} \quad \text{for all } x, y \text{ in } S \text{ with } x \neq y \text{ and } 0<\alpha<\frac{1}{2}.$$

Koparde and Waghmode [5] also extended this result to the pair of mappings T_1 and T_2 satisfying the Kannan type condition $\|T_1x-T_2y\|^2 \leq \alpha\left\{\|x-T_1x\|^2+\|y-T_2y\|^2\right\}$ for all x, y in S with $x \neq y$ and $0<\alpha<\frac{1}{2}$ and to their power p,q are some positive integers

In this paper we present a common fixed point theorem for rational inequality involving self mappings. For the purpose of obtaining the fixed point of the four continuous mappings. we have constructed a Cauchy sequence and have shown its convergence to the fixed point.

II. MAIN RESULT

Theorem . Let T_1, T_2 , T_3 and T_4 be four continuous self mapping of a closed subset S of a Hilbert space H satisfying

$$T_1T_4 = T_4T_1 \text{ and } T_2T_3 = T_3T_2 \qquad \ldots(1)$$

$$\left.\begin{aligned} T_1(H) &\subset T_3(H) \\ T_2(H) &\subset T_4(H) \end{aligned}\right\} \qquad \ldots(2)$$

$$\|T_1x-T_2y\| \le \alpha\left\{\frac{\|T_1x-T_3y\|^2+\|T_2y-T_4x\|^2}{\|T_1x-T_3y\|+\|T_2y-T_4x\|}\right\}+\beta\|T_4x-T_3y\| \quad \ldots(3)$$

where $\alpha, \beta \ge 0$ and $2\alpha+\beta<1.$

Then T_1, T_2, T_3 and T_4 has a unique common fixed point.

Proof: Let $x_0 \in S$ by (2) there exists $x_1 \in S$ such that $T_3x_1 = T_1x_0$ and for this point x_1 we can choose a point $x_2 \in S: T_2x_1 = T_4x_2$ and so on.

Continuing in this manner, we can choose a sequence $\{y_n\}$ in H such that

$$y_{2n} = T_3x_{2n+1} = T_1x_{2n}$$
$$y_{2n+1} = T_4x_{2n+2} = T_2x_{2n+1} \quad ; \; n = 0,1,2,3,...$$

From (3)

$$\|y_{2n}-y_{2n+1}\| = \|T_1x_{2n}-T_2x_{2n+1}\|$$
$$\le \alpha\left\{\frac{\|T_1x_{2n}-T_3x_{2n+1}\|^2+\|T_2x_{2n+1}-T_4x_{2n}\|^2}{\|T_1x_{2n}-T_3x_{2n+1}\|+\|T_2x_{2n+1}-T_4x_{2n}\|}\right\}+\beta\|T_4x_{2n}-T_3x_{2n+1}\|$$
$$\le \alpha\left\{\frac{\|y_{2n}-y_{2n+1}\|^2+\|y_{2n+1}-y_{2n-1}\|^2}{\|y_{2n}-y_{2n+1}\|+\|y_{2n+1}-y_{2n-1}\|}\right\}+\beta\|y_{2n-1}-y_{2n}\|$$
$$\le \alpha\|y_{2n+1}-y_{2n-1}\|+\beta\|y_{2n-1}-y_{2n}\|$$
$$\le \alpha\|y_{2n+1}-y_{2n}\|+\alpha\|y_{2n}-y_{2n-1}\|+\beta\|y_{2n-1}-y_{2n}\|$$

$$\Rightarrow (1-\alpha)\|y_{2n}-y_{2n+1}\| \le (\alpha+\beta)\|y_{2n-1}-y_{2n}\|$$
$$\Rightarrow \|y_{2n}-y_{2n+1}\| \le \frac{(\alpha+\beta)}{(1-\alpha)}\|y_{2n-1}-y_{2n}\|$$

If $k = \dfrac{(\alpha+\beta)}{(1-\alpha)} < 1 \qquad (\because 2\alpha+\beta<1)$

$$\Rightarrow \|y_{2n}-y_{2n+1}\| \le k\|y_{2n-1}-y_{2n}\| \quad \ldots(4)$$

Now $\|y_{2n-1}-y_{2n}\| = \|T_2x_{2n-1}-T_1x_{2n}\|$

$$= \|T_1x_{2n}-T_2x_{2n-1}\|$$

$$\leq \alpha \left\{ \frac{\|T_1x_{2n}-T_3x_{2n+1}\|^2+\|T_2x_{2n-1}-T_4x_{2n}\|^2}{\|T_1x_{2n}-T_3x_{2n+1}\|+\|T_2x_{2n+1}-T_4x_{2n}\|} \right\} + \beta\|T_3x_{2n-1}-T_4x_{2n}\|$$

$$\leq \alpha \left\{ \frac{\|y_{2n}-y_{2n-2}\|^2+\|y_{2n-1}-y_{2n-1}\|^2}{\|y_{2n}-y_{2n-2}\|+\|y_{2n-1}-y_{2n-1}\|} \right\} + \beta\|y_{2n-2}-y_{2n-1}\|$$

$$\leq \alpha\ \|y_{2n}-y_{2n-2}\| + \beta\|y_{2n-2}-y_{2n-1}\|$$

$$\Rightarrow (1-\alpha)\|y_{2n-1}-y_{2n}\| \leq (\alpha+\beta)\|y_{2n-2}-y_{2n-1}\|$$

$$\Rightarrow \|y_{2n-1}-y_{2n}\| \leq k\|y_{2n-2}-y_{2n-1}\| \qquad \ldots(5)$$

From (4) and(5)

$$\|y_{2n}-y_{2n+1}\| \leq k^2\|y_{2n-2}-y_{2n-1}\| \qquad \ldots(6)$$

Hence for $n = 0,1,2,3,\ldots$

$$\|y_n - y_{n+1}\| \leq k\|y_{n-1}-y_n\|$$

From (6)

$$\|y_n - y_{n+1}\| \leq k^n\|y_0-y_1\|$$

Now consider for positive integer, we have

$$\|y_n - y_{n+p}\| = \|y_n - y_{n+1} + y_{n+1} - y_{n+2} + y_{n+2} + \ldots y_{n+p--} - y_{n+p}\|$$

$$\leq \|y_n - y_{n+1}\| + \|y_{n+1} - y_{n+2}\| + \ldots + \|y_{n+p--} - y_{n+p}\|$$

$$\leq (\mathrm{k}^n + \mathrm{k}^{n+1} + \ldots + \mathrm{k}^{n+p-1})\|y_0 - y_1\|$$

$$= \frac{k^n}{1-k}\|y_0-y_1\| \to 0 \text{ as } \mathrm{n} \to \infty.$$

Therefore $\{y_n\}$ is a Cauchy's sequence in S i.e. in H and so sequence $\{y_n\}$ converges to some $u \in S$.

Therefore the sequences $\{T_3x_{2n+1}\},\{T_1x_{2n}\},\{T_4x_{2n+2}\}$ and $\{T_2x_{2n+1}\}$ are also converges to some *u*.

Now from (1) T_1, T_2, T_3 and T_4 are continuous for

$$T_1\{T_4x_n\} \to T_1u$$
$$T_4\{T_1x_n\} \to T_4u$$
$$T_2\{T_3x_n\} \to T_2u$$
$$T_3\{T_2x_n\} \to T_3u$$

From (1)

$$T_1u = T_4u$$
$$T_2u = T_3u$$

Now we will show that u is common fixed point of T_1 and T_4.

Consider

$$\begin{aligned}
\|T_1u-u\| &= \|T_1u-y_{2n+1}+y_{2n+1}-u\| \\
&\le \|T_1u-y_{2n+1}\|+\|y_{2n+1}-u\| \\
&\le \|T_1u-T_2x_{2n+1}\|+\|y_{2n+1}-u\| \\
&\le \alpha\left\{\frac{\|T_1u-T_3x_{2n+1}\|^2+\|T_2x_{2n+1}-T_4u\|^2}{\|T_1u-T_3x_{2n+1}\|+\|T_2x_{2n+1}-T_4u\|}\right\}+\beta\ \|T_3x_{2n+1}-T_4u\|+\|y_{2n+1}-u\| \\
&\le \alpha\|T_1u-y_{2n}\|+\alpha\|y_{2n+1}-T_1u\|+\|y_{2n+1}-u\|+\beta\|y_{2n}-T_1u\|+\|y_{2n+1}-u\| \\
&\le 2\alpha\|T_1u-u\|+\beta\|u-T_1u\|+\|u-u\| \text{ as } n\to\infty.
\end{aligned}$$

$$\Rightarrow \{1-(2\alpha+\beta)\}\ \|T_1u-u\| \le 0$$
$$\Rightarrow \quad \|T_1u-u\| = 0$$
$$\Rightarrow \quad T_1u = u$$

This gives $T_1u = T_4u = u$ i.e. u is fixed. Similarly we can show that $T_2\,u = T_3\,u = u$ so that u is common fixed point of T_1, T_2, T_3 and T_4.

Corollary:
Let T_1, T_2 and T_3 be three continuous self mapping of a closed subset S of a Hilbert space H satisfying

$$T_1T_3 = T_3T_1 \text{ and } T_2T_3 = T_3T_2$$

$$\left.\begin{aligned} T_1(H) &\subset T_3(H) \\ T_2(H) &\subset T_3(H) \end{aligned}\right\}$$

$$\|T_1x - T_2y\| \le \alpha\left\{\frac{\|T_1x - T_3y\|^2 + \|T_2y - T_3x\|^2}{\|T_1x - T_3y\| + \|T_2y - T_3x\|}\right\} + \beta\|T_3x - T_3y\| \quad \text{where } \alpha, \beta \ge 0 \text{ and } 2\alpha + \beta < 1.$$

$$\Rightarrow \quad \|T_1x - T_2y\| \le \alpha\{\|T_1x - T_3y\| + \|T_2y - T_3x\|\} \quad \text{where } \alpha \ge 0 \text{ and } 2\alpha < 1.$$

Then T_1, T_2, and T_3 has a unique common fixed point.

III. CONCLUSION

In this paper we replace Kannan inequality by rational inequality The theorems proved in this paper by using rational inequality is improved and stronger form of some earlier inequality given by Kannan [3], Koparde and Waghmode [5]. We have obtained a unique fixed point for four continuous self mapping satisfying rational inequality.

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