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On Common Fixed Point for Intimate Mapping in Fuzzy Metric Space

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ABSTRACT: In this paper, we proved some generalized common fixed point theorem for intimate mapping in fuzzy metric space. Our results generalize and extend some recent results in Common Fixed Point for Intimate Mappings. Examples and some applications are given to show the usability of the presented results.

Key words: Fixed point, common fixed point, Intimate mapping, compatible mapping of type (A).

I. INTRODUCTION

The concept of fuzzy sets introduced by Zadeh [18] in 1965 and after one decade the concept of fuzzy metric space is introduced by Kramosil and Michalek [7] in 1975.The notion of fuzzy metric spaces with the help of continuous t-norms is revised by George and Veeramani [2]. In 1976, common fixed point theorem for commuting maps is given by Jungck [5], which generalizes Banach's fixed point theorem. Commutativity is defined by Sessa [10] and he also proved common fixed point theorems for weakly commuting maps. Furthermore generalized commutativity introduced by Jungck [5], this property is also called compatibility, which is more general than that of weak commutativity. The concept of Compatible maps of type (A) in metric space and Banach space by motivating the concept of compatible maps is introduced by Jungck,Murthy and Cho [6]. The concept of Intimate mapping in metric space is generalized by Sahu, Dhagat and Shrivastava[11]. Fuzzy set theory has applications in applied sciences such as neural network theory, stability theory, and medical sciences. In this paper we proved a common fixed point theorem for intimate mapping in fuzzy metric space.

II. PRELIMINARIES

Definition 2.1 [18]. Let X be a non empty set. Let A fuzzy set A in X is a function with domain X and values [0, 1].

Definition 2.2 [13]. A binary operation $*$ on $[0,1]$ is called continuous t-norm satisfying the following conditions:

(i) $*$ is commutative and associative;

(ii) $*$ is continuous;

(iii) $a * 1 = a, \forall a \in [0,1]$;

(iv) $a * b \leq c * d$ Whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Definition 2.3 [15]. Suppose X is an arbitrary set, * is a continuous t-norm, M is a fuzzy set in X^2 x $[0,\infty)$ then the triplet (X, M,*) is called fuzzy metric space satisfying the following conditions

(i) $M(x, y, 0) = 0$ for all $x, y \in X$;

(ii) $M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ iff $x = y$;

(iii) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ $\forall x, y, z \in X$ and $s, t > 0$;

(v) $M(x, y, .): [0,\infty) \rightarrow [0,1]$ is left continuous for all $x, y \in X$;

(vi) $\lim_{n\rightarrow\infty} M(x, y, t) = 1$, for all $x, y \in X$.

Definition 2.4 [7]. In a fuzzy metric space (X, M,*), a sequence $\{x_n\}$ is called Cauchy sequence if $\lim_{n\rightarrow\infty} M(x_{n+p}, x_n, t) = 1$ for every $t > 0$ and each p> 0.

Definition 2.5 [7]. In a fuzzy metric space (X, M,*), a sequence $\{x_n\}$ is said to be convergent to $x \in X$ if $\lim_{n\rightarrow\infty} M(x_n, x, t) = 1$ for each $t > 0$.

Definition 2.6 [7] .A fuzzy metric space is called complete iff every Cauchy sequence in X converges in X.

Definition 2.7[9]. Two self mappings f and g of a fuzzy metric space (X, M,*) are said to be weakly commuting if $M(fgx, gfx, t) \geq M(fx, gx, t)$ for every $x \in X$.

Definition 2.8 [11]. Let A and S be two self mappings of fuzzy metric space X. If $\lim_{n\to\infty} M\,(ASx_n\,, SSx_n, t) = \lim_{n\to\infty} M\,(SAx_n\,, AAx_n, t) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$. Then {A, S} is said to be compatible pair of type (A).

Definition 2.9 [11]. Let A and S be two self mappings of fuzzy metric space X .Then {A, S} is said to be S-intimate if and only if $\alpha M(SAx_n\,, Sx_n, t) \le \alpha M(ASx_n\,, Ax_n, t)$

Where $\alpha = lim$ sup or lim inf, $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \epsilon X$.

Example : Let X=[0,1] and A,S are self mappings on X defined as follows AX $=\frac{2}{x+2}$ and SX$=\frac{1}{x+1}$ for all x in [0,1]. Now the sequence $\{x_n\}$ in X defined by $x_n = \frac{1}{n}, n\epsilon N$.Then we have $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 1$.Again $M\big(ASx_n\,, Ax_n, t\big) \to \frac{1}{3}$ and$M(SSx_n\,, Sx_n, t) \to \frac{1}{2}$.

As n→ ∞ Then clearly we have $\lim_{n\to\infty} M\,(ASx_n\,, Ax_n, t) \le \lim_{n\to\infty} M\,(SSx_n\,, Sx_n, t)$. Thus (A, S) is A – intimate. But {A, S} is not compatible mapping of type (A).

Preposition (I): The pair {A, S} is both A and S intimate If it is compatible of type (A).

Proof : Since $M(ASx_n, Ax_n, t) \le M(ASx_n, SSx_n, t) + M(SSx_n, Sx_n, t)$.

For nϵN therefore $\alpha M(ASx_n, Ax_n, t) \le \alpha M(ASx_n, SSx_n, t) + \alpha M(SSx_n, Sx_n, t)$

$\alpha M(ASx_n, Ax_n, t) \le \alpha M(SSx_n, Sx_n, t)$.

Whenever $\{x_n\}$ is a sequence in fuzzy metric space X such that $\lim_{n\to\infty} A\,x_n = \lim_{n\to\infty} S\,x_n = t$ for some tϵX, Thus the pair {A, S} is A- intimate.

Similarly we have shown that the pair {A, S} is S-intimate but its converse need not be true.

Preposition (II): Let A and S be two self mappings of a fuzzy metric space X .If the pair {A, S} is S-intimate and At=St=pϵX. Then $M(Sp, p, t) \le M(Ap, p, t)$.

Proof: Suppose $x_n = t$ for all $n \ge 1$, so $Ax_n = Sx_n = At = St = p$, Since the pair {A, S} is S-intimate then

$$M(SAt, St, t) = \lim_{n\to\infty} M(SA\,x_n . Sx_n, t)$$
$$\le \lim_{n\to\infty} M(AA\,x_n . Ax_n, t)$$
$$= M(AAt, At, t)$$
$$M(Sp, p, t) \le M(Ap, p, t).$$

Lemma (1) : (Singh and Meade 1977).$\gamma(t) < t$ for every t >0 if and only if $\lim_{n\to\infty} \gamma^n(t) = 0$ where γ^ndenotes the n times composition of γ.

We now suppose that A, B, S and T be self mappings of fuzzy metric space X such that

(1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$

(2) $M(AX, BY, t) \le r[\max\{M(Sx, Ty, t), M(Sx, Ax, t), M(Sx, By, t), M(Ty, By, t)\}]$ for all x, y in X and $0 < r < 1$.

Let x_0 be an arbitrary point in X. Then ∃ a point x_1 in X such that $Ax_0 = Tx_1$ (by (1)) and then a point x_2 in X such that $Bx_1 = Sx_2$ and so on.We obtain a sequence $\{y_n\}$ in X such that

(3) $y_{2n} = Sx_{2n} = Bx_{2n-1}$

$y_{2n+1=} Tx_{2n+1} Ax_{2n}$ for $n = 1,2,3\ldots\ldots\ldots\ldots$

Now prove the following Lemma:

Lemma (2): Let A, B, S and T be the self mappings in fuzzy metric space satisfying (1) and (2). Then the sequence $\{y_n\}$ is a Cauchy sequence as defined by (3).

Proof: Using (2) and (3) we get

$$M\big(y_{2n+1,} y_{2n,} t\big) = M(Ax_{2n,} Bx_{2n-1}, t)$$
$$\le r[\max\{\, M\big(Sx_{2n,} Tx_{2n-1}, t\big), M\big(Sx_{2n,} Ax_{2n}, t\big), M\big(Sx_{2n,} Bx_{2n-1}, t\big), M\big(Tx_{2n-1,} Bx_{2n-1}, t\big)\}]$$
$$= r[\max\{\, M\big(y_{2n,} y_{2n-1}, t\big), M\big(y_{2n,} y_{2n+1}, t\big), M\big(y_{2n,} y_{2n}, t\big), M\big(y_{2n,} y_{2n-1}, t\big)\}]$$

If $M(y_{2n+1}, y_{2n}, t) > M(y_{2n}, y_{2n-1}, t)$

Then $M(y_{2n+1}, y_{2n}, t) \le M(y_{2n+1}, y_{2n}, t)$ which is a contradiction.

Thus $M(y_{2n+1}, y_{2n}, t) \le rM(y_{2n}, y_{2n-1}, t)$.

Similarly $M(y_{2n+2}, y_{2n+1}, t) \le rM(y_{2n+1}, y_{2n}, t)$.

Now $M(y_{n+1}, y_n, t) \le rM(y_n, y_{n-1}, t)$.

$$\le r^2 M(y_{n-1}, y_{n-2}, t).$$

....
$\leq r^n M(y_1, y_0, t)$.

For every integer k>0, we get

$M(y_n, y_{n+k}, t) \leq M(y_n, y_{n+1}, t) + M(y_{n+1}, y_{n+2}, t) \ldots \ldots + M(y_{n+k-1}, y_{n+k}, t)$
$\leq (1 + r + r^2 + \cdots \ldots \ldots \ldots r^{k-1}) M(y_n, y_{n+1}, t)$.
$\leq \left\{\frac{r^k}{(1-r)}\right\} M(y_n, y_{n+1}, t) \to 0 \; as \; n \to \infty$.

Thus $M(y_n, y_{n+k}, t) \to 0$.

Therefore $\{y_n\}$ is a Cauchy sequence .Now we shall prove our main result .

Theorem : Let A, B, S and T be self mappings in a fuzzy metric space satisfying (1), (2), (3) and following condition

(4)The pair {A, S} is S-intimate and {B, T} is T-intimate.

(5)S(X) is complete.

Then A, B, S and T have a unique common fixed point in X.

Proof : Suppose the sequence $\{y_n\}$ (as above by a lemma 2) is a Cauchy sequence and has a limit u in X.Since S(X) is complete and $\{Sx_{2n}\}$is a Cauchy sequence. Then the sequence converges to a point p=Su for some u in X. Also $\{Ax_{2n}\}$, $\{Bx_{2n-1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ are subsequences of $\{y_n\}$,this subsequence converge to p. Hence $x_{2n}, Sx_{2n}, Bx_{2n-1}$ and $Tx_{2n+1} \to p$.Now from (2) we get

$M(Au, Bx_{2n+1}, t) \leq r[\max\{M(Su, Tx_{2n+1}, t), M(Su, Au, t), M(Su, Bx_{2n+1}, t), M(Tx_{2n+1,} Bx_{2n+1}, t)\}]$ as $n \to \infty$.
$M(Au, p, t) \leq r[\max\{M(Su, p, t), M(Su, Au, t), M(Su, p, t), M(p, p, t)\}]$.
$M(Au, p, t) \leq r[\max\{M(p, p, t), M(p, Au, t), M(p, p, t), M(p, p, t)\}]$.
$M(Au, p, t) \leq rM(p, Au, t)$.

Which is contradiction, so $Au = p = Su$.

Since $A(X) \subseteq T(X), \exists v \epsilon X$ such that $Tv = p$. Hence for (2) we have

$M(p, Bv, t) = M(Au, Bv, t) \leq r[\max\{M(Su, Tv, t), M(Su, Au, t), M(Su, Bv, t), M(Tv, Bv, t)\}]$.
$M(p, Bv, t) \leq rM(p, Bv, t)$.

Which is a contradiction so $Bv = p = Tv$.

Since $Au = p = Su$ and the pair $\{A, S\}$ is s-intimate .Then by preposition (2) we have

$M(Sp, p, t) \leq M(Ap, p, t)$

Suppose $Ap \neq p$ then from (2) we get

$M(Ap, p, t) = M(Ap, Bv, t) \leq r[\max\{M(Sp, Tv, t), M(Sp, Ap, t), M(Sp, Bv, t), M(Tv, Bv, t)\}]$
$= r[\max\{M(Sp, p, t), M(Sp, Ap, t), M(Sp, p, t), M(p, p, t)\}]$
$\leq r[\max\{M(Ap, p, t), M(Ap, p, t), M(p, Ap, t), M(Ap, p, t), M(p, p, t)\}]$
$\leq rM(Ap, p, t)$.

Which is a contradiction, so $Ap = p$ and also $= p$.

Hence $Ap = Sp = p$.

Similarly we get $Bp = Tp = p$.

Uniqueness : Let us consider q is another common fixed point of A,B,S and T such that $p \neq q$,therefoe

$M(p, q, t) = M(Ap, Bq, t) \leq r[\max\{M(Sp, Tq, t), M(Sp, Ap, t), M(Sp, Bq, t), M(Tq, Bq, t)\}]$
$= r[\max\{M(p, q, t), M(p, p, t), M(p, q, t), M(q, q, t)\}]$
$\leq rM(p, q, t)$. which is a contradiction .This shows that p=q.

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