International Journal of Theoretical & Applied Sciences, Special Issue-NCRTAST 8(1): 139-141(2016)

ISSN No. (Print): 0975-1718
ISSN No. (Online): 2249-3247

Pseudo-S- Metric Spaces and Pseudo-S-Metric Product Spaces

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(Received 11 April, 2016 Accepted 20 May, 2016)
(Published by Research Trend, Website: www.researchtrend.net)

ABSTRACT: Shaban Sedghi and Nguyen Van Dung [2] prove a general fixed point theorem in S-metric spaces which is a generalization of S. Sedghi, N. Shobe, A. Aliouche, Mat. Vesnik [3] as applications, they get many analogues of fixed point theorems from metric spaces to S-metric spaces. Inspired by their work In this paper we define pseudo-S- metric spaces and pseudo-S-metric product spaces.

Keywords: S- metric spaces, pseudo-S- metric spaces and pseudo-S-metric product spaces.

I. INTRODUCTION

We begin with the following definition:
DEFINITION (1): Let X be a nonempty set. An S-metric on X is a function S: $X^3 \rightarrow [0,\infty)$ that satisfies the following conditions, for each x, y, z ∈X

1) S (x, y, z) ≥ 0
2) S (x, y, z) = 0 if and only if x=y=z
3) S (x, y, z) ≤ S (x, x, a) + S (y, y, a) + S (z, z, a)

The pair (X, S) is called an S-metric space.
Immediate examples of such S-metric spaces are:
EXAMPLE 1.1 Let X = R^n and $\|.\|$ a norm on X, then S(x, y, z) = $\|$x-z $\|$ + $\|$ y-z $\|$ is an S-metric on X
EXAMPLE 1.2 Let X = R^n and $\|.\|$ a norm on X, then S(x, y, z) = $\|$y+z-2x $\|$ + $\|$ y-z $\|$ is an S-metric on X
EXAMPLE 1.3 Let X be a nonempty set, d is ordinary metric on X, then S(x, y, z) = d(x, z)+ d(y; z) is an S-metric on X.
DEFINITION 2: Let S: X x X x X → R with the following properties ∀ x, y, z ∈ X

1) S(x, y, z) ≥ 0
2) If x=y=z then S(x, y, z) =0
3) S (x, y, z) ≤ S (x, x, a) + S (y, y, a) + S (z, z, a)

then the function S is called a pseudo-S-metric, on X , and the pair (X, S) is a pseudo- S-metric space.
Theorem 1.1: Every S-metric space is a pseudo S-metric but every pseudo S-metric is not necessarily S-metric space.
Proof: From definition of S-metric space it is obvious that every S-metric space is pseudo S-metric space to proof converse let us define
S: R x R x R → R^+ s.t.
S (x, y, z) = $|x^2$-$z^2|$ + $|y^2$-$z^2|$ ∀ x, y, z ∈ X
It is easy to check that S is pseudo S-metric on R^+ evidently.
S (x, y, z) = 0 ⇒ x^2-z^2=0 and y^2-z^2=0
⇒x=±z and y=±z
Thus S (x, y, z) = 0 does not necessarily implies x=y and y=z and z=x
Hence (R, S) is pseudo S-metric but not S-metric space.
Let $\{X_i, S_i\}$: i=1,2,3…..n} be a collection of pseudo S-metric Spaces and X denote the product of the sets X=$\prod_i X_i$it is natural to ask whether or not it is possible to define a pseudo S-metric on X.
The next theorems give a positive answer to this question.In the case of finite or a denumerable collection of pseudo S-metric space.

II. MAIN RESULT

Theorem 2.1(2): If (X_1,S_1), (X_2,S_2), (X_3,S_3)....... (X_m,S_m) be a pseudo S-metric and let x= $(a_1, a_2, a_3, \ldots, a_m)$, y=$(b_1, b_2, , b_3, \ldots, b_m)$ and z= $(c_1, c_2, c_3, \ldots c_m)$ are arbitrary points in product set X=$\prod_i X_i$ then each of the following functions:
(i) S (x, y, z) = $[S_1\ (a_1, b_1, c_1)^2 + S_2\ (a_2, b_2, c_2)^2 + \ldots + S_m\ (a_m, b_m, c_m)^2]^{1/2}$
(ii) S (x, y, z) = Max $\{S_1\ (a_1, b_1, c_1), S_2\ (a_2, b_2, c_2), \ldots, S_m\ (a_m, b_m, c_m\}$
(iii) S (x, y, z) = $S_1\ (a_1, b_1, c_1) + S_2\ (a_2, b_2, c_2) + \ldots + S_m\ (a_m, b_m, c_m)$
are pseudo S-metric on X .known as the product pseudo S-metric on X and (x, s) is known as pseudo S-metric product space.
Proof let S(x.y.z) = $[S_1\ (a_1, b_1, c_1)^2 + S_2\ (a_2, b_2, c_2)^2 + \ldots + S_m\ (a_m, b_m, c_m)^2]^{1/2}$
. i = 1, 2, ….m ⇒ $S_i\ (a_i, b_i, c_i) \geq 0$ ∀ i = 1, 2, …. m

$\sum_{i=1}^{m} Si\ (ai, bi, ci)\ )^2 = 0, \forall\ i = 1, 2, \ldots m$

$\Rightarrow S\ (x, y, z) \geq 0$ …(1)

Let $S\ (x,y,z) = \{\ [S_1\ (a_1, b_1, c_1)^2 + S_2\ (a_2, b_2, c_2)^2 + \ldots + S_m\ (a_m, b_m, c_m)^2]^{1/2}$

If x, y and z are equal $\Rightarrow a_i, b_i, c_i$ are equal for all i=1,,2,3……..m

$\sum S_i\ (a_i, b_i, c_i)^2 = 0$

$\Rightarrow S\ (x, y, z) = 0$ …(2)

$s(x,y,z) \leq [\ S_1\ (a_1, b_1, c_1)^2 + S_2\ (a_2, b_2, c_2)^2 + \ldots + S_m\ (a_m, b_m, c_m)^2]^{1/2}$

$\leq [[\ S_1\ (a_1, a_1, k_1) + S_1\ (b_1, b_1, k_1) + S_1(c_{1,}, c_1, k_1]^2 +$

$S_2\ (a_2, a_2, k_2) + S_2\ (b_2, b_2, k_2) + S_2(c_{2,}, c_2, k_2]^2 +$

$S_3\ (a_3, a_3, k_3) + S_3\ (b_3, b_3, k_3) + S_3(c_{3,}, c_3, k_3]^2 +$

……………………………………..+

$S_m\ (a_m, a_m, k_m) + S_m\ (b_m, b_m, k_m) + S_m(c_{m,}, c_m, k_m]^2]^{1/2}$

$[\ \sum [S_i(a_i, a_i, k_i) + S_i\ (b_i, b_i, k_i) + S_i(c_{i,}, c_i, k_i]^2]^{1/2},$

$\forall\ i = 1,2,3\ldots m$

$\leq [\ \sum [S_i(a_i, a_i, k_i)^2]^{1/2} + [\ \sum S_i\ (b_i, b_i, k_i)^2]^{1/2} + [\ \sum S_i(c_{i,}, c_i, k_i]^2]^{1/2},$

$\forall\ i = 1,2,3\ldots m$ (by Minkowski's inequality)

$= s(x, x, k) + s(y, y, k) + s(z, z, k)$ …(3)

now let $S\ (x, y, z) = \max\{S_1\ (a_1, b_1, c_1) + S_2\ (a_2, b_2, c_2) + \ldots + S_m\ (a_m, b_m, c_m)\}$

$\forall\ i = 1,2,3\ldots m$

$\Rightarrow \max\ (S_i\ (a_i, b_i, c_i) \geq 0\ \forall\ i = 1, 2, \ldots m$

$\Rightarrow S(x,y,z) \geq 0\ \forall\ i = 1, 2, \ldots m$ …(4)

If x, y and z are equal $\Rightarrow a_i, b_i, c_i$ are equal for all i=1,,2,3……..m

$\Rightarrow S_i\ (a_i, b_i, c_i) = 0$

$\max\{S_1\ (a_1, b_1, c_1) + S_2\ (a_2, b_2, c_2) + \ldots + S_m\ (a_m, b_m, c_m)\} = 0$

$\Rightarrow S\ (x, y, z) = 0$ …(5)

let $k = (k_1, k_2, k_3 \ldots k_m) \in X$, Then

$S(x,y,z) = S_1\ (a_1, b_1, c_1) + S_2\ (a_2, b_2, c_2) + \ldots + S_m\ (a_m, b_m, c_m)\}$

$S(x,x,k) = \max\{S_1\ (a_1, a_1, k_1) + S_2\ (a_2, a_2, k_2) + \ldots + S_m\ (a_m, a_m, k_m)\}$

$S(y, y, k) = \max\{S_1\ (b_1, b_1, k_1) + S_2\ (b_2, b_2, k_2) + \ldots + S_m\ (b_m, b_m, k_m)\}$

$S(z,z,k) = \max\{S_1\ (c, c_1, k_1) + S_2\ (c_2, c_2, k_2) + \ldots + S_m\ (c_m, c_m, k_m)\}$

Now, $S\ (x, y, z) = \max\{S_1\ (a_1, b_1, c_1) + S_2\ (a_2, b_2, c_2) + \ldots + S_m\ (a_m, b_m, c_m)\}$

$\leq \max\{S_1\ (a_1, a_1, k_1) + S_2\ (a_2, a_2, k_2) + \ldots + S_m\ (a_m, a_m, k_m)\} + \max\{S_1\ (b_1, b_1, k_1) + S_2\ (b_2, b_2, k_2) + \ldots + S_m\ (b_m, b_m, k_m)\} + \max\{S_1\ (c, c_1, k_1) + S_2\ (c_2, c_2, k_2) + \ldots + S_m\ (c_m, c_m, k_m)\}$

$\leq S(x, x, k) + S(y, y, k) + S\ (z, z, k)$ …(6)

Let $S\ (x, y, z) = S_1\ (a_1, b_1, c_1) + S_2\ (a_2, b_2, c_2) + \ldots + S_m\ (a_m, b_m, c_m)$

$\forall\ i = 1, 2, \ldots m \Rightarrow S_i\ (a_i, b_i, c_i) \geq 0$

$\sum S_i\ (a_i, b_i, c_i) = 0, \forall\ i = 1, 2, \ldots m$

$\Rightarrow S\ (x, y, z) \geq 0$ …(7)

Let $S\ (x,y,z) = S_1\ (a_1, b_1, c_1) + S_2\ (a_2, b_2, c_2) + \ldots + S_m\ (a_m, b_m, c_m)$

If x, y and z are equal $\Rightarrow a_i, b_i, c_i$ are equal for all i=1,,2,3……..m

$\sum S_i\ (a_i, b_i, c_i) = 0, \forall\ i = 1, 2, \ldots m$

$\Rightarrow S\ (x, y, z) = 0$ …(8)

$S\ (x,y,z) = S_1\ (a_1, b_1, c_1) + S_2\ (a_2, b_2, c_2) + \ldots + S_m\ (a_m, b_m, \square_\square)$

$\leq \{\square_1\ (\square_1, \square_1, \square_1) + \square_2\ (\square_2, \square_2, \square_2) + \ldots + \square_\square\ (\square_\square, \square_\square, \square_\square)\} + \{\square_1\ (\square_1, \square_1, \square_1) + \square_2\ (\square_2, \square_2, \square_2) + \ldots + \square_\square\ (\square_\square, \square_\square, \square_\square)\} + \{\square_1\ (\square, \square_1, \square_1) + \square_2\ (\square_2, \square_2, \square_2) + \ldots + \square_\square\ (\square_\square, \square_\square, \square_\square)\}$

$\leq S(x, x, k) + S(y, y, k) + S\ (z, z, k)$ …(9)

By the given equations it is proved that every S-metric space is a pseudo S-metric but every pseudo S-metric is not necessarily S-metric space.

Theorem 2.2

If $\{(\square_1, \square_1), (\square_2, \square_2), (\square_3, \square_3) \ldots \ldots (\square_\square, \square_\square)\}$ be denumerable collection of pseudo s metric space and let $x = (\square_1, \square_2, \square_3, \ldots, \square_\square)$, $y = (\square_1, \square_2, \square_3, \ldots, \square_\square)$ and $z = (\square_1, \square_2, \square_3, \ldots \square_\square)$ are arbitrary points in product set $X = \prod_\square \square_\square$ then the function s defined by

$S(x,y,z) = \sum_{\square=1}^{\infty} \frac{1}{2^\square} \left(\frac{\square_\square(\square_\square, \square_\square, \square_\square,)}{1 + \square_\square(\square_\square, \square_\square, \square_\square,)}\right)$ is a pseudo s metric on x

Proof 1. $S_n\ (a_n, b_n, c_n) \geq 0\ \forall\ n = 1, 2, 3\ldots, \infty$

$\Rightarrow S(x,y,z) \geq 0\ \forall\ n = 1, 2, 3\ldots\infty$.

2. If x, y and z are equal $\Rightarrow a_n, b_n, c_n$ are equal for all n=1,,2,3……..∞

$\Rightarrow S_n\ (a_n, b_n, c_n) = 0\ \forall\ n = 1, 2, 3\ldots, \infty$

$\Rightarrow S\ (x, y, z) = 0\ \forall\ n = 1, 2, 3\ldots, \infty$

3. Let $k = (k_1, k_2, k_3 \ldots, k_m) \in X$ then

$$\frac{1}{2^\square}\left(\frac{\square_\square(\square_\square, \square_\square, \square_\square,)}{1 + \square_\square(\square_\square, \square_\square, \square_\square,) + \square_\square(\square_\square, \square_\square, \square_\square,)\square_\square(\square_\square, \square_\square, \square_\square,)}\right.$$

$$\leq \frac{s_n(a_n, a_n, k_n,)}{1 + s_n(a_n, a_n, k_n,)}$$

$$\frac{1}{2^n}\frac{s_n(b_n, b_n, k_n,)}{1 + s_n(a_n, a_n, k_n,) + s_n(b_n, b_n, k_n,)s_n(c_n, c_n, k_n,)}$$

And

$$\leq \frac{s_n(b_n, b_n, k_n,)}{1 + s_n(b_n, b_n, k_n,)}$$

$$\frac{1}{2^n}\frac{s_n(c_n, c_n, k_n,)}{1 + s_n(a_n, a_n, k_n,) + s_n(b_n, b_n, k_n,)s_n(c_n, c_n, k_n,)}$$

$$\leq \frac{s_n(c_n, c_n, k_n,)}{1 + s_n(c_n, c_n, k_n,)}$$

Now S (x,y,z)

$$= \sum_{n=1}^{\infty} \frac{1}{2^n} (\frac{s_n(a_n, b_n, c_n,)}{1 + s_n(a_n, b_n, c_n,)}$$

Now S (x,y,z)

$$\leq \frac{1}{2^n}\left(\frac{s_n(a_n, a_n, k_n,)}{1 + s_n(a_n, a_n, k_n,) + s_n(b_n, b_n, k_n,)s_n(c_n, c_n, k_n,)}\right)$$

$$+ \frac{1}{2^n}\left(\frac{s_n(b_n, b_n, k_n,)}{1 + s_n(a_n, a_n, k_n,) + s_n(b_n, b_n, k_n,)s_n(c_n, c_n, k_n,)}\right)$$

$$+ \frac{1}{2^n} (\frac{s_n(c_n, c_n, k_n,)}{1 + s_n(a_n, a_n, k_n,) + s_n(b_n, b_n, k_n,)s_n(c_n, c_n, k_n,)}$$

It shows that

$$\frac{1}{2^n} (\frac{s_n(a_n, b_n, c_n,)}{1 + s_n(a_n, b_n, c_n,)}) \geq 0$$

It completes the proof.

REFERENCES

[1]. Lipschutz, S., Schaum's outline of theory and problem of General Topology (1965).

[1]. Fixed point theorems on s-metric spaces Shaban Sedghi and Nguyen Van Dzung. matematiqki vesnik **66**, 1 (2014), 113–124 March 2014.

[3]. S. Sedghi, N. Shobe, A. Aliouche, Mat. Vesnik A generalization of fixed point theorems in S-metric spaces **64** (2012), 258–266