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Common Fixed Point Theorems in Ordered Cone Metric Spaces

Shweta Maheshwari
**Department of Mathematics,*
……………………………………….Bhopal, (Madhya Pradesh), India

(Corresponding author: Shweta Maheshwari, shwetajiya10@gmail.com*)*
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ABSTRACT: In this paper we extend some existing results and prove fixed point theorem on partially ordered cone metric spaces which satisfy certain weak contractive inequalities.
In this consequence we have also given some illustrative examples.

Key Words: Partially ordered set; Cone metric space; Weak contractive inequality; Control function; Fixed point.
AMS Subject Classification: 54H10, 54H25

I. INTRODUCTION

Generalization of metric space has been introduced as cone metric space where every pair of elements is assigned to an element of a Banach space equipped with a cone which induces a natural partial order [16]. Other fixed point theorems in cone metric spaces were deduced in several other recently published works, some of which are noted in [14, 17, 19, 20].

Recently fixed point theory has developed rapidly in partially ordered metric spaces; that is, metric spaces endowed with a partial ordering. Earlier a result in this direction was established by Turinici in ordered metrizable uniform spaces [28]. Later Ran and Reurings established a fixed point result in partially ordered metric spaces and applied it to solve certain matrix equations [25].

Several other more recent works in this area are noted in [1, 15, 23].

Generalization of Banach's contraction principle is weak contraction principle which was first given by Alber et al. in Hilbert spaces [2] and subsequently extended to metric spaces by Rhoades [26].

In several works like [4, 12, 13, 29] fixed point problems involving weak contractions and mappings satisfying weak contraction type inequalities have been considered . Particularly, in cone metric spaces the weak contraction principle was extended by the present authors [9]. The use of control function in fixed point theory was initiated by Khan et al. [21] which they called Altering distance function. This function has been used in obtaining fixed point results in metric spaces [5, 22, 27] and probabilistic metric spaces [7, 8].

In this paper we prove some fixed point results in cone metric spaces having a partial order by using a control function. Precisely, we show that certain functions will have fixed points if they satisfy certain weak contractive inequalities. Our results extend the results of Altun *et al.* [3] in the special ordered cone metric spaces where the cone metric $d(x, y)$ for $x \neq y$, is constrained to the interior of the cone.

II. DEFINITIONS

Definition 2.1([16]) Let E always be a real Banach space and P a subset of E. P is called a cone if and only if:

(i) P is nonempty, closed, and $P \neq \{0\}$,
(ii) $a, b \in R$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$,
(iii) $x \in P$ and $-x \in P \Rightarrow x = 0$.

Given a cone $P \subset E$, a partial ordering $\leq$ with respect to P is naturally de fined by $x \leq y$ if and only if $y - x \in P$, for $x, y \in E$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$, where intP denotes the interior of P.

The cone P is said to be normal if there exists a real number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \Rightarrow \|x\| \leq K\|y\|.$$

The least positive number K satisfying the above statement is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent; that is, if $\{x_n\}$ is a sequence such that

$$x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots \leq y,$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$.

Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone. In the following we always suppose that E is a real Banach space with cone P in E with $\text{int}P \neq \phi$ and $\leq$ is the partial or dering with respect to P.

Definition 2.2 A function $\psi : P \to P$ is called an Altering distance function if the following properties are satisfied:

(i) ψ is strongly monotone increasing and continuous,

(ii) $\psi(t) = 0$ if and only if $t = 0$.

Definition 2.3 ([16]) Let X be a nonempty set. Let the mapping $d : X \times X \to E$ satisfies

(i) $0 \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$, for all $x, y \in X$,

(iii) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.4 ([17]) Let (X, d) be a cone metric space, $f : X \to X$ and $x_0 \in X$. Then the function f is continuous at x_0 if for any sequence $\{x_n\}$ in X, $x_n \to x_0$ implies $fx_n \to fx_0$.

Definition 2.5 ([3]) Let $(X, \leq)$ be a partially ordered set. Two mappings $f, g : X \to X$ are said to be weakly increasing if

$fx \preccurlyeq gfx$ and $gx \preccurlyeq fgx$ hold for all $x \in X$.

Definition 2.6 ([15]) Let $(X, \leq)$ be a partially ordered set and $T : X \longrightarrow X$ be a self map. We say that T is monotone nondecreasing if $x, y \in X$, $x \preccurlyeq y \Rightarrow Tx \preccurlyeq Ty$.

Lemma 2.1 ([16]) Let (X, d) be a cone metric space, P be a normal cone and $\{x_n\}$ be a sequence in X. Then:

(i) $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$,

(ii) $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

Lemma 2.2 Let E be a real Banach space with cone P in E. Then

(i) if $a \leq b$ and $b \ll c$, then $a \ll c$ [19],

(ii) if $a \ll b$ and $b \ll c$, then $a \ll c$ [19],

(iii) if $0 \leq x \leq y$ and $a \geq 0$, where a is real number, then $0 \leq ax \leq ay$ [19],

(iv) if $0 \leq x_n \leq y_n$, for $n \in N$ and $\lim x_n = x$, $\lim y_n = y$, then $0 \leq x \leq y$ [19],

P is normal if and only if $x_n \leq y_n \leq z_n$ and $\lim x_n = \lim z_n = x$ imply $\lim y_n = x$ [11].

Lemma 2.3 ([10]) Let (X, d) be a cone metric space with regular cone P such that $d(x, y) \in \text{int}P$, for $x, y \in X$ with $x \neq y$. Let $\varphi : \text{int}P \cup \{0\} \to \text{int}P \cup \{0\}$ be a function with the following properties:

(i) $\varphi(t) = 0$ if and only if $t = 0$,

(ii) $\varphi(t) \ll t$, for $t \in \text{int}P$ and

(iii) either $\varphi(t) \leq d(x, y)$ or $d(x, y) \leq \varphi(t)$, for $t \in \text{int}P \cup \{0\}$ and $x, y \in X$.

Let $\{x_n\}$ be a sequence in X for which $\{d(x_n, x_{n+1})\}$ is monotonic decreasing.

Then $\{d(x_n, x_{n+1})\}$ is convergent to either $r = 0$ or $r \in$ int P.

II. MAIN RESULTS

Lemma 3.1. Let (X, d) be a cone metric space. Let $\varphi : \text{int}P \cup \{0\} \longrightarrow \text{int}P \cup \{0\}$ be a function such that (i) $\varphi(t) = 0$ if and only if $t = 0$ and

(ii) $\varphi(t) \ll t$, for $t \in \text{int}P$.

Then a sequence $\{x_n\}$ in X is a Cauchy sequence if and only if for every $c \in E$ with $0 \ll c$ there exists $n_0 \in N$ such that $d(x_n, x_m) \ll \varphi(c)$, for all $n, m > n_0$.

Proof. Suppose that $\{x_n\}$ is a Cauchy sequence. Then for every $c \in E$ with $0 \ll c$ there exists $n_0 \in N$ such that $d(x_n, x_m) \ll c$, for all $n, m > n_0$. Let $c \in E$ with $0 \ll c$ be arbitrary.

By condition (i) of the lemma $\varphi(c) \in$ int P; that is, $0 \ll \varphi(c)$. Therefore, there exists $n_0 \in N$ such that $d(x_n, x_m) \ll \varphi(c)$, for all $n, m > n_0$.

Conversely suppose that for every $c \in E$ with $0 \ll c$ there exists n0 $\in N$ such that $d(x_n, x_m) \ll \varphi(c)$, for all $n, m > n_0$. Since $c \in \text{int}P$, by condition (ii) of the lemma, we have $\varphi(c) \ll c$.

Combining the above two inequalities by using the property (ii) of lemma 2.2, we obtain $d(x_n, x_m) \ll c$, for all $n, m > n_0$.

Therefore, for every $c \in E$ with $0 \ll c$ there exists $n_0 \in N$ such that $d(x_n, x_m) \ll c$, for all $n, m > n_0$.

Hence $\{x_n\}$ is a Cauchy sequence.

Theorem 3.1 Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a cone metric d in X for which the cone metric space (X, d) is complete with regular cone P such that $d(x, y) \in \text{int}P$, for $x, y \in X$ with $x \neq y$. Let $f : X \longrightarrow X$ be a continuous and non decreasing mapping with respect to $\preccurlyeq$ satisfying

$\psi(d(fx, fy)) \leq \psi(M(x, y)) - \varphi(d(x, y))$, for all $x, y \in X$ with $y \preccurlyeq x$, (3.1)

where

$M(x, y) = p\, d(x, y) + q\, [d(x, fx) + d(y, fy)] + r\, [d(x, fy) + d(y, fx)]$, with

$p, q, r \geq 0$, $p + 2q + 2r \leq 1$, and $\psi : P \to P$ and $\varphi : \text{int}P \cup \{0\} \to \text{int}P \cup \{0\}$

are continuous functions with the following properties:

(i) ψ is strongly monotonic increasing,

(ii) $\psi(t) = 0 = \varphi(t)$ if and only if $t = 0$,

(iii) $\varphi(t) \ll t$, for $t \in \text{int}P$ and

(iv) either $\varphi(t) \leq d(x, y)$ or $d(x, y) \leq \varphi(t)$, for $t \in \text{int}P \cup \{0\}$ and $x, y \in X$.

If there exists $x_0 \in X$ such that $x_0 \preccurlyeq fx_0$, then f has a fixed point in X.

Proof. If $fx_0 = x_0$, then the proof is completed. Suppose that $fx_0 \neq x_0$. Since $x_0 \preccurlyeq fx_0$ and f is nondecreasing w.r.t. $\leq$, we construct the sequence $\{x_n\}$ such that $x_n = fx_{n-1} = f^n x_0$ and $x_0 \leq fx_0 \leq f^2 x_0 \ldots \leq f^n x_0 \leq f^{n+1} x_0 \leq \ldots$; that is, $x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq x_{n+1} \leq \ldots$

Clearly, $x_n \leq x_{n+1}$, for each $n \geq 1$. Putting $x = x_{n+1}$ and $y = x_n$ in (3.1) we have

$\psi(d(x_{n+2}, x_{n+1})) = \psi(d(fx_{n+1}, fx_n))$
$\leq \psi(M(x_{n+1}, x_n)) - \varphi(d(x_{n+1}, x_n))$
$= \psi(p\ d(x_{n+1}, x_n) + q\ [d(x_{n+1}, x_{n+2}) + d(x_{n+1}, x_n))]$
$+ r\ [d(x_{n+1}, x_{n+1})) + d(x_n, x_{n+2})\ d(x_n, x_{n+2})] - \varphi(d(x_{n+1}, x_n)))$.

Since $d(x_n, x_{n+2})) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})$and ψ is strongly monotonic increasing,
it follows that

$\psi([d(x_{n+2}, x_{n+1})) \leq \psi(p\ [d(x_{n+1}, x_n)+ q\ [[d(x_{n+1}, x_{n+2})+ [d(x_n, x_{n+1}))]$
$+ r\ [[d(x_n, x_{n+1})+ [d(x_{n+1}, x_{n+2})]) - \varphi([d(x_{n+1}, x_n))$. (3.2)

Using a property of φ, we have

$\psi([d(x_{n+2}, x_{n+1})) \leq \psi(p\ [d(x_{n+1}, x_n)+ q\ [[d(x_{n+1}, x_{n+2})+ [d(x_n, x_{n+1}))]$
$+ r\ [[d(x_n, x_{n+1})+ [d(x_{n+1}, x_{n+2})])$

Using the strongly monotone property of ψ, we have

$d(x_{n+2}, x_{n+1}) \leq p\ [d(x_{n+1}, x_n)+ q\ [[d(x_{n+1}, x_{n+2})+ [d(x_n, x_{n+1}))]$
$+ r\ [d(x_n, x_{n+1})+ d(x_{n+1}, x_{n+2})]$

that is,

$(1 - q - r)\ d(x_{n+2}, x_{n+1}) \leq (p + q + r)\ d(x_{n+1}, x_n)$

that is,

$d(x_{n+2}, x_{n+1}) \leq (p+q+r)\ (1-q-r)\ d(x_{n+1}, x_n)$,

which implies that

$d(x_{n+2}, x_{n+1}) \leq d(x_{n+1}, x_n)$, (since $(p+q+r) / (1-q-r) \leq 1$).

Therefore, $\{ d(x_{n+1}, x_n) \}$is a monotone decreasing sequence. Hence by lemma 2.3, there exists u ▯ P with either $u = 0$ or u ▯ int P such that

$d(x_{n+1}, x_n) \to u$ as $n \to \infty$. (3.3)

Taking $n \to \infty$ in (3.2), using (3.3) and the continuities of ψ and φ, we have

$\psi(u) \leq \psi((p + 2q + 2r)u) - \varphi(u)$, which implies that

$\psi(u) \leq \psi(u) - \varphi(u)$, (since $p + 2q + 2r \leq 1$ and ψ is strongly monotonic increasing),

which is a contradiction unless $u = 0$.

Hence, $d(x_{n+1}, x_n) \to 0$ as $n -\to \infty$. (3.4)

Next we show that {xn} is a Cauchy sequence. If {xn} is not a Cauchy sequence, then by lemma 3.1, there exists a c ▯ E with 0 ▯ c, such that ▯ n_0 ▯ N, ▯ n, m ▯ N with $n > m \geq n_0$ such that

$d(x_n, x_m) < \neq \varphi(c)$. Hence by a property of φ in (iv) of the theorem, $\varphi(c) \leq d(x_n, x_m)$

Therefore, there exist sequences $\{m(k)\}$ and $\{n(k)\}$ in N such that for all positive integers k,

$n(k) > m(k) > k$ and $d(x_{nk}, x_{mk}) \geq \varphi(c)$.

Assuming that n(k) is the smallest such positive integer, we get $d(x_{nk}, x_{mk}) \geq \varphi(c)$

Now, $\varphi(c) \leq d(x_{nk}, x_{mk}) \leq d(x_{nk}, x_{m(k-1)}) + d(x_{m(k-1)}, x_{nk})$ that is,

$\varphi(c) \leq d(x_{nk}, x_{mk}) \leq d(x_{nk}, x_{m(k-1)}) + \varphi(c)$.

Letting $k \to \infty$ in the above inequality, using (3.4) and the property (v) of Lemma 2.2,
we have

$\lim k \to \infty\ d(x_{nk}, x_{mk}) = \varphi(c)$. (3.5)

Again,

$d(x_{nk}, x_{mk}) \leq d(x_{nk}, x_{m(k+1)})) + d(x_{n(k+1)}, x_{m(k+1)}) + d(x_{mk}, x_{m(k)})$

and $d(x_{nk}, x_{m(k+1)}) \leq d(x_{n(k+1)}, x_{nk}) + d(x_{nk}, x_{mk}) + d(x_{mk}, x_{m(k+1)})$

Letting $k \to \infty$ in above inequalities, using (3.4) and (3.5), we have

$\lim k \to \infty\ d(x_{n(k+1)}, x_{m(k+1)}) = \varphi(c)$. (3.6)

Again,

$d(x_{nk}, x_{m(k+1)}) \leq d(x_{nk}, x_{mk}) + d(x_{mk}, x_{m(k+1)})$

and

$d(x_{nk}, x_{mk}) \leq d(x_{nk}, x_{m(k+1)}) + d(x_{mk}, x_{m(k+1)})$

Letting $k \to \infty$ in the above four inequalities, using (3.4) and (3.5), we have

$\lim k \to \infty\ d(x_{nk}, x_{m(k+1)}) = \varphi(c)$, (3.7)

$\lim k \to \infty\ d(x_{n(+1)k}, x_{mk}) = \varphi(c)$. (3.8)

Using (3.4), (3.5), (3.7) and (3.8), we have

$\lim k \to \infty\ M(x_{nk}, x_{mk}) = \lim k \to \infty\ [p\ d(x_{nk}, x_{mk})\ ) + q\ (d(x_{nk}, x_{n(k+1)})) + d(x_{mk}, x_{m(k+1)}))$
$+ r\ d(x_{nk}, x_{m(k+1)}) + d(x_{mk}, x_{n(k+1)})] = (p + 2r)\ \varphi(c)$. (3.9)

Clearly, $x_m(k) \leq x_n(k)$. Putting $x = x_n(k)$, $y = x_m(k)$ in (3.1), we have

$\psi\ d(x_{nk}, x_{m(k+1)}) = \psi(d(fx_{nk},f\ x_{mk})\ ) \leq \psi(M(x_{nk}, x_{mk})\ ) - \varphi(d(x_{nk}, x_{mk})\ )$.

Letting $k \to \infty$ in the above inequality, using (3.5), (3.6), (3.9) and the continuities of ψ and φ, we have $\psi(\varphi(c)) \leq \psi((p + 2r)\ \varphi(c)) - \varphi(\varphi(c))$, that is,

$\psi(\varphi(c)) \leq \psi(\varphi(c)) - \varphi(\varphi(c))$, (since $p+ 2r \leq 1$ and ψ is strongly monotonic increasing),

which is a contradiction by virtue of a property of φ. Hence $\{x_n\}$ is a Cauchy sequence.

From the completeness of X, there exists z ▯ X such that $x_n \to z$ as $n \to \infty$. (3.10)

Since f is continuous and $x_n \to z$ as $n \to \infty$, $\lim n \to \infty\ fx_n = fz$, that is,

$\lim n \to \infty\ x_{n+1} = fz$, that is, $= fz$.

Hence z is a fixed point of f and the proof is completed.

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