International Journal of Theoretical & Applied Sciences, 12(1): 11-16(2020)

ISSN No. (Print): 0975-1718
ISSN No. (Online): 2249-3247

Analysis of Small-deflection in a Thermoelastic plate

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(Received 04 May, 2020, accepted 18 June, 2020)
(Published by Research Trend, Website: www.researchtrend.net)

ABSTRACT: This present article deals with the study of transient thermoelastic problem of a plate, in which we determine the temperature distribution and thermal deflection on upper plane surface of a thick clamped square/rectangular plate, when the interior second kind condition is known in context of fractional order theory of thermoelasticity. The integral transform techniques are used to find the solution.

Keywords: Square/rectangular plate, Fourier and Laplace transform Thermal deflection.

I. INTRODUCTION

Thermoelasticity based on the heat conduction equation with differential operators of fractional order. Time-fractional differential operators describe memory effects, space-fractional differential operators deal with the long-range interaction. It should be emphasized that fractional calculus has been successfully used in physics, geology, chemistry, rheology, engineering, bioengineering, robotics, etc. Various mathematical aspects of fractional calculus can be found in Oldham and Spanier [4], in the remarkably comprehensive encyclopedic-type treatise by Samko *et al.* [5], Miller and Ross [6], Podlubny [3] and Diethelm [13] devoted to fractional differential equations, and in the recent in-depth monograph by Kilbas *et al.* [2], Gorenflo and Mainardi [1] and Valerio *et al.* [14]). A related progress in thermoelasticity recorded from Lamba *et al.* [7-12]. Kumar and Kamdi [18] investigated thermal behavior of a finite hollow cylinder in context of fractional thermoelasticity with convection boundary conditions. Lamba and Kamdi [19] determined temperature and thermal stress functions for functionally graded materials cylinder. Lamba and Khobragade [20-23] studied some thermoelastic problem within the context of fractional thermoelasticity.

In the present problem an attempt is made to study the transient thermoelastic problem to determine the unknown temperature gradient, temperature distribution and thermal deflection of the plate occupying the space *D:* $\{(x, y, z) \in R^3 : 0 \le x \le a, 0 \le y \le b, 0 \le z \le h\}$ with the known interior heat flux . Geometry of the plate is given in Fig.1. Finite Fourier sine transform and Laplace transform techniques are used to find the solution of the problem. Numerical estimate for the temperature distribution on upper plane surface is obtained.

II. STATEMENT OF THE PROBLEM

Consider a thick isotropic rectangular plate occupying the space *D.* The differential equation satisfied by the deflection $w(x, y, t)$, is

$$D\nabla^4 w(x, y, t) = \frac{-\nabla^2 M_T(x, y, t)}{1-\nu} \tag{1}$$

where ν is the Poisson's ratio of the plate material, M_T denote the thermal momentum of the plate and D denote the flexural rigidity,

where $\nabla^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2}$

and the resultant thermal momentum M_T is defined as

$$M_T(x, y, t) = \alpha E \int_0^h z T(x, y, z, t) dz \tag{2}$$

where α, E are the coefficient of linear expansion, Young's modulus respectively. Since the edge of the circular plate is fixed and clamped,

$$w = \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial y^2} = 0 \text{ at } x = 0, a \text{ and } y = 0, b \tag{3}$$

The temperature of the plate at time t satisfying the differential equation

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{k}\frac{\partial^\alpha T}{\partial t^\alpha} \tag{4}$$

where k is the thermal diffusivity of the material of the plate,

In Equation (4), $\frac{\partial^\alpha T}{\partial t^\alpha}$ is the Caputo fractional derivative [9]-[11] as

$$\frac{d^\alpha f(t)}{dt^\alpha}=\begin{cases}\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\frac{d^n f(\tau)}{d\tau^n}d\tau, & n-1<\alpha<n,\\ \frac{d^n f(\tau)}{dt^n}, & \alpha=n\end{cases} \tag{5}$$

with the following Laplace transform rule

$$L\left\{\frac{d^\alpha f(t)}{dt^\alpha}\right\}=s^\alpha L\{\bar{f}(s)\}-\sum_{k=0}^{n-1} f^{(k)}(0^+)\,s^{\alpha-1-k}, \quad n-1<\alpha<n. \tag{6}$$

in which s is the transform parameter.
subject to the initial and boundary conditions

$$T(x,y,z,0)=0 \tag{7}$$

$$[T(x,y,z,t)]_{x=0}=0 \tag{8}$$

$$[T(x,y,z,t)]_{x=a}=0 \tag{9}$$

$$[T(x,y,z,t)]_{y=0}=0 \tag{10}$$

$$[T(x,y,z,t)]_{y=b}=0 \tag{11}$$

$$\left[\frac{\partial T(x,y,z,t)}{\partial z}\right]_{z=0}=0 \tag{12}$$

$$\left[\frac{\partial T(x,y,z,t)}{\partial z}\right]_{z=h}=g(x,y,t) \tag{13}$$

Eqns. (1) to (13) constitute the mathematical formulation of the problem under consideration.

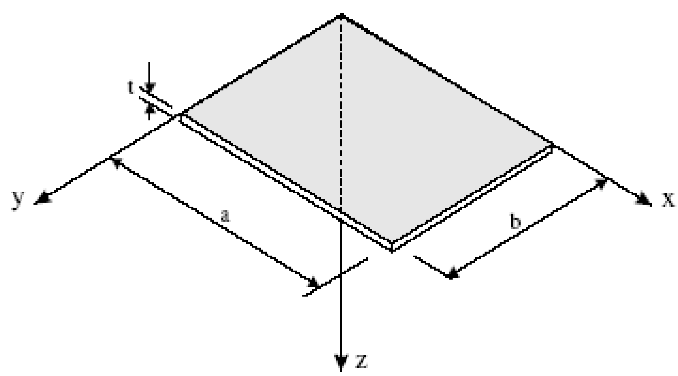

Fig. 1. Geometry of three dimensional Plate.

III. SOLUTION OF THE PROBLEM

By applying finite Fourier sine transform on the coordinate x, y and the Laplace transform and their inverses to the Eqns. (4) to (13), one obtains the expressions for the unknown temperature gradient and temperature distribution respectively are as

$$g(x,y,t)=\frac{8k}{ab\xi}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{l+1}\lambda_l\sin(\lambda_l h)\sin(px)\sin(qy)$$
$$\times\int_0^t \bar{\bar{f}}(m,n,t')\,E_\alpha\left\{-k\left[p^2+q^2+\lambda_l^{\,2}\right](t-t')\right\} \tag{14}$$

$$T(x,y,z,t) = \frac{8k}{ab\xi}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^l \cos(\lambda_l z)\sin(px)\sin(qy)$$

$$\times\int_0^t \overline{\overline{f}}(m,n,t')\,E_\alpha\left\{-k\left[p^2+q^2+\lambda_l^2\right](t-t')\right\} \tag{15}$$

where *l, m, n* are the positive integers and $\lambda_l = \frac{l\pi}{\xi}$, $p = \frac{m\pi}{a}$, $q = \frac{n\pi}{b}$,

$$L^{-1}\left[\frac{1}{s^k + k\left[p^2+q^2+\lambda_l^2\right]}\right] = E_\alpha\left\{-k\left[p^2+q^2+\lambda_l^2\right](t-t')\right\} \tag{16}$$

IV. DETERMINATION OF THERMAL DEFLECTION

Substituting the value of $T(x, y, z, t)$ from equation (15) in equation (2) one obtains

$$M_T(x,y,t) = \frac{8k\alpha E}{ab\xi}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^l\left(\int_0^h z\cos(\lambda_l z)dz\right)\sin(px)\sin(qy)$$

$$\times\int_0^t \overline{\overline{f}}(m,n,t')\,E_\alpha\left\{-k\left[p^2+q^2+\lambda_l^2\right](t-t')\right\} \tag{17}$$

We assume that the solution of Eqn. (1) satisfying Eqn. (3) as

$$w(x,y,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} w_{mn}\sin(px)\sin(qy) \tag{18}$$

Using the equations (17) and (18) in (1), one obtains

$$w_{mn}(t) = \frac{8k\alpha E}{D(1-\nu)ab\xi}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^l\left(\frac{\lambda_l h\sin(\lambda_l h)+\cos(\lambda_l h)-1}{\lambda_l^2(p^2+q^2)}\right)$$

$$\times\int_0^t \overline{\overline{f}}(m,n,t')E_\alpha\left\{-k\left[p^2+q^2+\lambda_l^2\right](t-t')\right\} \tag{19}$$

Substituting the value of $w_{mn}(t)$ in Eqn. (18), one obtains the expression for thermal deflection as

$$w(x,y,t) = \frac{8k\alpha E}{D(1-\nu)ab\xi}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^l\left(\frac{\lambda_l h\sin(\lambda_l h)+\cos(\lambda_l h)-1}{\lambda_l^2(p^2+q^2)}\right)\sin(px)\sin(qy)$$

$$\times\int_0^t \overline{\overline{f}}(m,n,t')E_\alpha\left\{-k\left[p^2+q^2+\lambda_l^2\right](t-t')\right\} \tag{20}$$

V. SPECIAL CASE

Set $f(x,y,t) = (1-e^{-t})(x^2-ax)(y^2-by)$ (21)

Applying finite Fourier sine transform w. r. to x, y to the equation (21) one obtains

$$\overline{\overline{f}}(m,n,t) = \frac{4(1-e^{-t})\left[(-1)^m-1\right]\left[(-1)^n-1\right]}{p^3q^3} \tag{22}$$

Substituting the value of (22) in the Eqn. (14), (15) and (20) one obtains

$$g(x,y,t) = \frac{32k}{ab\xi}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{l+1}\left[(-1)^m-1\right]\left[(-1)^n-1\right]\left(\frac{\lambda_l}{p^3q^3}\right)\sin(\lambda_l h)\sin(px)\sin(qy)$$

$$\times\int_0^t (1-e^{-t'})\left\{-k\left[p^2+q^2+\lambda_l^2\right](t-t')\right\} \tag{23}$$

$$T(x,y,z,t)=\frac{32k}{ab\xi}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{l}\left[(-1)^{m}-1\right]\left[(-1)^{n}-1\right]\left(\frac{1}{p^{3}q^{3}}\right)\cos(\lambda_{l}z)\,\sin(px)\,\sin(qy)$$

$$\times\int_{0}^{t}(1-e^{-t'})\left\{-k\left[p^{2}+q^{2}+\lambda_{l}^{2}\right](t-t')\right\} \tag{24}$$

$$w(x,y,t)=\frac{16k\alpha E}{D(1-\nu)ab\xi}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{l}\left[(-1)^{m}-1\right]\left[(-1)^{n}-1\right]\left(\frac{\lambda_{l}h\sin(\lambda_{l}h)+\cos(\lambda_{l}h)-1}{\lambda_{l}^{2}p^{3}q^{3}(p^{2}+q^{2})}\right)$$

$$\times\sin(px)\sin(qy)$$

$$\times\int_{0}^{t}(1-e^{-t'})\left\{-k\left[p^{2}+q^{2}+\lambda_{l}^{2}\right](t-t')\right\} \tag{25}$$

VI. NUMERICAL RESULTS

Numerical calculations have been carried out for the Aluminum plate.

Set $\beta=\frac{32k}{ab\xi}$, $\gamma=\frac{16k\alpha E}{D(1-\nu)ab\xi}$, a = 1, b = 2, ξ=0.75, h =2, t = 1 sec and k= 0.86 and $\alpha=25.5\times10^{-6}$, $E=6.9\times10^{11}$ in equations (23) to (25) one obtains

$$\frac{g(x,y,t)}{\beta}=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{l+1}\left[(-1)^{m}-1\right]\left[(-1)^{n}-1\right]\left(\frac{\lambda_{l}}{p^{3}q^{3}}\right)\sin(\lambda_{l}h)\,\sin(px)\,\sin(qy)$$

$$\times\int_{0}^{t}(1-e^{-t'})\left\{-k\left[p^{2}+q^{2}+\lambda_{l}^{2}\right](t-t')\right\} \tag{26}$$

$$\frac{T(x,y,z,t)}{\beta}=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{l}\left[(-1)^{m}-1\right]\left[(-1)^{n}-1\right]\left(\frac{1}{p^{3}q^{3}}\right)\cos(\lambda_{l}z)\,\sin(px)\,\sin(qy)$$

$$\times\int_{0}^{t}(1-e^{-t'})\left\{-k\left[p^{2}+q^{2}+\lambda_{l}^{2}\right](t-t')\right\} \tag{27}$$

$$\frac{w(x,y,t)}{\gamma}=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{l}\left[(-1)^{m}-1\right]\left[(-1)^{n}-1\right]\left(\frac{\lambda_{l}h\sin(\lambda_{l}h)+\cos(\lambda_{l}h)-1}{\lambda_{l}^{2}p^{3}q^{3}(p^{2}+q^{2})}\right)\sin(px)\sin(qy)$$

$$\times\int_{0}^{t}(1-e^{-t'})\left\{-k\left[p^{2}+q^{2}+\lambda_{l}^{2}\right](t-t')\right\} \tag{28}$$

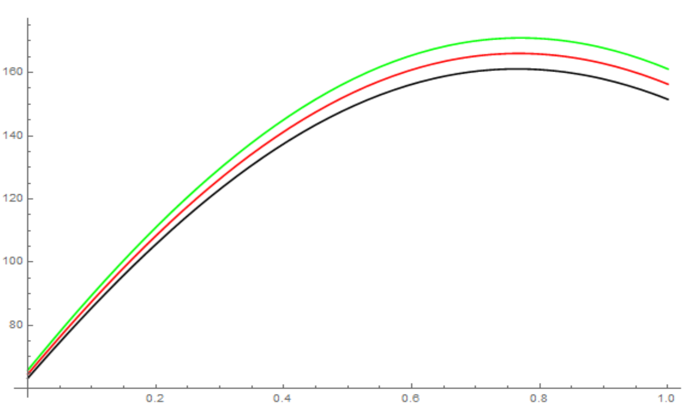

Fig. 2. Graph of temperature distribution function verses time.

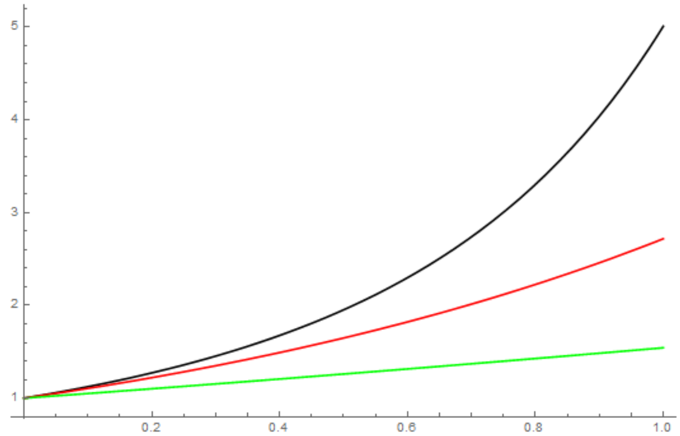

Fig. 3. Graph of thermal deflection function verses time.

VII. CONCLUSION

The unknown temperature gradient, temperature distribution and thermal deflection on upper plane surface of a thick rectangular plate have been obtained in context of fractional order theory of thermoelasticity, when the interior second kind condition and the other boundary conditions are known; with the aid of finite Fourier sine transform and Laplace transform techniques. The results are obtained in the form of infinite series. The series solutions converge provided we take sufficient number of terms in the series.

The expressions for temperature and deflection are represented graphically. The temperature distribution and deflection that are obtained can be applied to the design of useful structures or machines in engineering applications.

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