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A Note on Separation Theorem and Continuous Linear Functionals

Neha Phogat
Resource Person, Department of Mathematics,
Maharshi Dayanand University, Rohtak (Haryana), India.

(Corresponding author: Neha Phogat)
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ABSTRACT: In this article, we obtain a depiction of continuous linear functionals on a fuzzy quasi-normed space, and indicate the firm of all continuous linear functional forms a convex cone. Finally, we establish a theorem of separation and Hahn-Banach for convex subsets.

Keywords: fuzzy quasi-normed space, continuous linear functional separation theorem

I. INTRODUCTION

Alegre and Romaguera [2] formulated problem using fuzzy quasi-norm, while [1] obtained the properties of the paratopological vector spaces that are quasi-metrizable, locally bounded, quais-normable. In [4], they established some results in fuzzy quasi-normed spaces. The [3] was expanded upon by [4] by proving an extension theorem for continuous linear functionals on a fuzzy normed space.

This paper consists of four sections. Section 1 contains introduction. Section 2 consists of basic definitions and prepositions. In section 3, we discuss continuous linear functionals on a "fuzzy quasi-normed space". In section 4, we prove Hahn-Banach and separation theorems for convex subsets.

II. PRELIMINARIES

Definition 1 [8]: A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if it satisfies the following conditions: $\forall a, b, c, d \in [0,1]$,

(1) $a * b = b * a$ (commutavity);

(2) $(a * b) * c = a * (b * c)$ (associativity);

(3) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ (monotonicity);

(4) $a * 1 = a$ (boundary condition);

(5) $*$ is continuous on $[0,1] \times [0,1]$ (continuity).

Three paradigmatic examples of continuous t-norm are $\wedge, \cdot$ and $*_L$ (the Lukasiewicz t-norm), which are defined by

$a \wedge b = \min\{a, b\}$, $a \cdot b = ab$ and $*_L\, b = \max\{a + b - 1, 0\}$, respectively.

Definition 2 [2]: A fuzzy quasi-norm on a real vector space X is a pair $(N,*)$ such that * is a continuous t-norm and N is a fuzzy set $X \times [0, +\infty)$ satisfying the following conditions: for every $, y \in X$,

(FQN1) $N(x, 0) = 0$;

(FQN2) $N(x,t) = N(-x,t) = 1$ for all $> 0 \Leftrightarrow x = \theta$;

(FQN3) $N(\lambda x, t) = N(x, t/\lambda)$ for all > 0 ;

(FQN4) $N(x,t) * N(y,s) \leq N(x + y, t + s)$ for all $, s > 0$;

(FQN5) $N(x, \_): [0, +\infty] \to [0,1]$ is left continuous;

(FQN6) $\lim_{t\to\infty} N(x,t) = 1$.

Obviously, the function $N(x, \_)$ is increasing for each $x \in X$.

By a fuzzy quasi-normed space, we mean a triple $(X, N,*)$ such that X is a real vector space and $(N,*)$ is a fuzzy quasi-norm on X.

If condition (FQN6) is omitted, we say that $(N,*)$ is a weak fuzzy quasi-norm on X.

Each fuzzy quasi-norm $(N,*)$ on X induces a T_o topology τ_n on X which has a basen given by the family of open balls

$\mathcal{B}(x) = \{B_n(x,r,t): r \in (0,1), t > 0\}$

at $x \in X$,

where,

$B_n(x,r,t) = \{y \in X: N(y - x, t) > 1 - r\}$.

We denote $cl_N A$ the closure of A and by $int_N A$ the interior of A in the topological space (X, τ_n).

A subset A of a real vector space X is

(1) Semi-balanced [7] provided that for each $x \in A$, $rx \in A$ whenever $0 \leq r \leq 1$;

(2) absorbing provided that for each $x \in X$, there is $\lambda_o > 0$ such that $\lambda_o x \in A$.

Remark 2.1. Obviously, we have

(1) if A is semibalanced, then A is absorbing if and only if for each $x \in X$, there is $\lambda_o > 0$ such that $\lambda x \in A$ whenever $0 < \lambda < \lambda_o$;

(2) if $\theta \in A$ and A is convex, then A is semibalanced.

Proposition 2.1 [2]. Let$(X, N,*)$ be a fuzzy quasi-normed space and let $\mathcal{B}(\theta)$ the family of open balls with center in the origin θ. Then:

(1) $B_N(\theta,r,t)$ is absorbing for all $t > 0$ and $r \in (0,1)$.

(2) $B_N(\theta,r,t)$ is semi-balanced for all $t > 0$ and $r \in (0,1)$.

(3) $\lambda B_N(\theta,r,t) = B_N(\theta,r,\lambda t)$ for every $\lambda > 0, t > 0$ and $r \in (0,1)$.

(4) If$U \in \mathcal{B}(\theta)$, there is $V \in \mathcal{B}(\theta)$, such that $V + V \subseteq U$.

(5) If$U, V \in \mathcal{B}(\theta)$, there is $W \in \mathcal{B}(\theta)$, such that $W \subseteq U \cap V$.

(6) $\forall x \in X, x + B_N(\theta, r, t) = B_N(x, r, t)$.

Remark 2.2. If the continuous t-norm * is chosen as "$\wedge$", then each element $\mathcal{B}(\theta)$ is convex.

Remark 2.3. By Proposition 2.1, the mappings: $(x, y) \rightarrow x + y$ and $(\lambda, x) \rightarrow \lambda x$ are continuous on $X \times X$ and $[0, \infty) \times X$, respectively, and the topology τ_n is translation invariant.

Proposition 2.2 ([2]). If $(X, N, *)$ is a fuzzy quasi-normed space, then $(X, \tau_n, *)$ is a quasi-metrizable paratopological vector space.

Proposition 2.3. Let $P = \{p_\alpha: p_\alpha$ is a function from X to $[0, \infty), \alpha \in (0,1)\}$ be a family of star quasi-seminorms. For each $x \in X$, let

$$U_p(x) = \{U(x: \alpha_1, \alpha_2, \dots, \alpha_n; \varepsilon): \varepsilon > 0; \alpha_1, \alpha_2, \dots, \alpha_n \in (0,1), n \in \mathbb{N}\},$$

where

$$U(x: \alpha_1, \alpha_2, \dots, \alpha_n; \varepsilon) = \{y \in X: p_{\alpha_i}(y - x) < \varepsilon, \alpha_i \in (0,1), i = 1,2, \dots, n\}$$

$$= \bigcap_{i=1}^{n} \{y \in X: p_{\alpha_i}(y - x) < \varepsilon, \alpha_i \in (0,1)\}$$

$$= \{y \in X: p_{\max\{\alpha_i: 1 \le i \le n\}}(y - x) < \varepsilon$$

Then, $U_p(x)$ is a basis of neighbourhoods of x.

III. CONTINUOUS LINEAR FUNCTIONALS ON A "FUZZY QUASI-NORMED SPACE

Consider the quasi-norm $w(x_1) = \max\{x_1, 0\}$ on the real numbers $\mathbb{R}$. The topology $\tau(w)$ generated by w is called the upper topology of $\mathbb{R}$. A basis of open $\tau(w)$-neighbourhoods of a point $x_1 \in \mathbb{R}$ is formed of the intervals $(-\infty, x_1 + \varepsilon), \varepsilon > 0$.

The quasi-dual $(X_1, N, *)^\#$ of a fuzzy quasi-normed space $(X_1, N, *)$ is formed by all continuous linear functionals from (X_1, τ_N) to $(\mathbb{R}, \tau(w))$. In the sequel, $(X_1, N, *)^\#$ will be simply denoted by $X_1^\#$.

Theorem 3.1 Let $(X_1, N, *)$ be a fuzzy quasi-normed space. $f \in X_1^\#$ iff there are $\alpha \in (0,1)$ and $M_1 > 0$ $s.t. h(x_1) \le M_1 ||x_1||_\alpha$ for all $x_1 \in X_1$.

Corollary 3.1 Let $(X_1, N, *)$ be a fuzzy quasi-normed space. $(X_1, N, *)^\#$ is a convex cone.

Now, we shall equip $(X_1, N, *)^\#$ with a weal fuzzy quasi-norm.

Definition 3.1 Let X_1 be a linear space and let $q_1: X_1 \rightarrow [0, \infty]$ be an extended function $\forall\, i \in I$. If $[q_1: i \in I]$ fulfils the conditions of star quasi-seminorms, then it is called a family of star extended quasi-seminorms.

Theorem 3.2 Let $Q = \{||.||_\alpha: \alpha \in (0,1)\}$ be an increasing family of separating star extended quasi-seminorms on real linear space X_1, and let $||.||_o$ be given by $||x||_o = 0\ \forall x_1 \in X_1$.The function $N_q(x_1, t): X_1 \times [0, \infty] \rightarrow [0,1]$ is given by

$$N_q(x_1, t) = \begin{cases} 0, & t = 0 \\ \sup\{\alpha \in (0,1): ||x_1||_\alpha < t, & t > 0 \end{cases} \quad (3.1)$$

Then $(N_q, *)$ is a weak fuzzy quasi-norm on X_1.

(FQN1) is obvious.

(FQN2) If $N_q(x_1, t) = N_q(-x_1, t)\ \forall\, t > 0$ then $||x_1||_\alpha < t$ and $||-x_1||_\alpha < t\ \forall\, \alpha \in (0,1)$ from (3.1). Therefore, $||x_1||_\alpha = ||-x_1||_\alpha = 0\ \forall\, \alpha \in (0,1)$. Since Q is separating, $x_1 = \theta$. Conversely, if $x_1 = \theta$, then it implies that $||x_1||_\alpha = ||-x_1||_\alpha = 0\ \forall\, t > 0$.

By (3.1), $N_q(x_1, t) = N_q(-x_1, t) = 1$.

(FQN3) Let $d > 0$. From $(*\, QN1)$, we have

$$N_q(dx_1, t) = \sup\left\{\alpha \in (0,1): ||dx_1||_\alpha < t\right\}$$

$$= \sup\left\{\alpha \in (0,1): ||x_1||_\alpha < \frac{t}{c}\right\}$$

$$= N_q(x_1, t/c).$$

(FQN4) Let $x_1, y_1 \in X_1$ and $s, t > 0$ and let $N_q(x_1, t) = \beta, N_q(y_1 s) = \gamma$. W.L.O.G., we assume that $0 < \min\{\beta, \gamma\}$.

For any $0 < \epsilon < \min\{\beta, \gamma\}$, there exist $\alpha', \alpha'' \in (0,1)\ s.t. \alpha' > \beta - \epsilon, \alpha'' > \gamma - \epsilon, ||x_1||_{\alpha'} < t$ and $||y_1||_{\alpha''} < s$.

Thus, $||x_1||_{\beta-\epsilon} < t$ and $||y_1||_{\gamma-\epsilon} < s$. And hence, $||x_1 + y_1||_{(\beta-\epsilon)*(\gamma-\epsilon)} \le ||x_1||_{\beta-\epsilon} + ||y_1||_{\gamma-\epsilon} < t + s$.

By (3.1), $N_q(x_1 + y_1, t + s) \ge (\beta - \epsilon) * (\gamma - \epsilon)$.

(FQN5) Obviously, $N_q(\theta, \_) = 1$, and hence, $N_q(\theta, \_)$ is continuous. Now, take $x_o \in X/\{\theta\}$ and $t_o > 0$.

If $N_q(x_1, t_o) = 0$, then $N_q(x_1, t) = N_q(x_1, t_0) = 0\ \forall\, t < t_o$.

So, $N_q(x_1, \_)$ is left continuous at t_o. Take $\epsilon > 0$, from (3.1), $\exists\, \alpha_o \in (0,1)\ s.t. ||x_1||_{\alpha_o} < t_o$ and $N_q(x_o, t) - \epsilon < \alpha_o$. So, we have $N_q(x, t) \ge \alpha_o\ \forall\, t$ with $||x_1||_{\alpha_o} < t < t_o$. Hence, $N_q(x, t_o) - N_q(x_1, t) \le N_q(x_1, t_o) - \alpha_o < \epsilon$.

Therefore, $N_q(x_1, \_)$ is left continuous at t_o. And

$$||h||_\alpha^\# = \sup\left\{h(x_1): ||x_1||_{1-\alpha} \le 1\right\} \forall\, \alpha \in (0,1). \quad (3.2)$$

Theorem 3.3 Let $(X_1, N, *)$ be a fuzzy quasi-normed space, $h \in X^\#, \alpha \in (0,1)$.

1. If $h \ne 0$, then $||h||_\alpha^\# > 0$.
2. $||h||_\alpha^\# = \sup\left\{h(x_1): ||x_1||_{1-\alpha} < 1\right\}$.
3. $||h||_\alpha^\# = \sup\{h(x_1): N(x_1, 1) \ge 1 - \alpha\}$.
4. If $N(x_1, \_)$ is increasing strictly, then $||h||_\alpha^\# = \sup\{h(x_1): N(x_1, 1) \ge 1 - \alpha\}$

Theorem 3.4 Let $(X_1, N, *)$ be a fuzzy quasi-normed space. Then

(1) $\{||.||_\alpha^\#: \alpha \in (0,1)\}$ is a family of separating star extended quasi-seminorms on $X_1^\#$;

(2) $\{||.||_\alpha^\#: \alpha \in (0,1)\}$ is increasing with respect to $\alpha \in (0,1)$.

Remark 3.1 $||f||_\alpha^\#$ can be infinity even in symmetrical situations [3].

The following theorem is obvious from theorem 3.2 and theorem 3.4.

Theorem 3.5 Let $(X_1, N,*)$ be a fuzzy quasi-normed space. For each $h \in X_1^{\#}$, let

$$N_{x_1}^{\#}(h,t) = \begin{Bmatrix} 0, t = 0 \\ \sup\{\alpha \in [0,1]: ||h||_{\alpha}^{\#} < t \end{Bmatrix} \quad (3.3)$$

Then, $(N_{x_1}^{\#},*)$ is a weak fuzzy quasi-norm on $X_1^{\#}$.

IV. HAHN-BANACH AND SEPARATION THEOREMS FOR CONVEX SETS

Lemma 4.1 Let X_1 be a linear space and q be a sublinear functional on X_1. If X_o is a subspace of X_1 and h_o is a linear functional by q on X_o, then $\exists$ a h dominated by q on X_1s.t.$\frac{h}{X_o} = h_o$.

Theorem 4.1 Let $(X_1, N,*)$ be a fuzzy quasi-normed space and let h_o be a continuous linear functional on a subspace $(X_o, N/X_o,*)$ of $(X_1, N,*)$. Then, $\exists\ \delta \in [0,1]$ for which the following two conditions are satisfied:

(1) for all$\alpha \in (0,\delta)$, there is $h_\alpha \in (X_1, N,*)^{\#}$s.t. $\frac{h_\alpha}{X_o} = h_o$ and $||h_\alpha||_{\alpha}^{\#} = ||h_o||_{\alpha,X_o}^{\#}$,where

$$||h_o||_{\alpha,X_o}^{\#} = \sup\left\{h_o(x_1): x_1 \in X_o, ||x_1||_{1-\alpha} \le 1\right\};$$

(2) $N_{X_o}^{\#}(h_o,t) = \sup\{N^{\#}(h_\alpha,t): \alpha \in (0,\delta)\}\ \forall\ t > 0$.

Proof:

Put

$$\delta = \sup\{\alpha \in (0,1): ||h_o||_{\alpha,X_o}^{\#} < \infty\} \quad (4.1)$$

Since $h_o \in (X_o, N/X_o,*)^{\#}$, we get $\delta \in (0,1)$.

(1) For any $\alpha \in (0,\delta)$, (4.1) implies that $||h_o||_{\alpha,X_o}^{\#} < \infty$.

Define a functional q_α on X_1 as:

$$q_\alpha(x) = ||h_o||_{\alpha,X_o}^{\#} ||x_1||_{1-\alpha}, \forall x_1 \in X_1.$$

$||.||_{1-\alpha}$is a quasi-seminorm implying that q_α is a sublinear functional on X.

Let $x_1 \in X_0$. If $||x_1||_{1-\alpha} > 0$, then $h_o(\frac{x_1}{||x_1||_{1-\alpha}}) \le ||h_o||_{\alpha,X_o}^{\#}$ so that $h_o(x_1) \le q_\alpha(x)$.

If $||x_1||_{1-\alpha} = 0$, then$||\varsigma x_1||_{1-\alpha} = \varsigma ||x_1||_{1-\alpha} = 0\ \forall\ \varsigma > 0$.

By definition of $||h_o||_{\alpha,X_o}^{\#}$, we get

$||h_o||_{\alpha,X_o}^{\#} > h_o(\varsigma x_1)$ i.e. $h_o(x_1) \le ||h_o||_{\alpha,X_o}^{\#}/\varsigma$

$\Rightarrow h_o(x_1) \le 0 = q_\alpha(x)$.

Thus, h_o is dominated by q_α on X_o.

By lemma 4.1, there is a linear functional h_α on X, s.t. $\frac{h_\alpha}{X_o} = h_o$ and

$h_\alpha(x_1) \le ||h_o||_{\alpha,X_o}^{\#} ||x_1||_{1-\alpha}, \forall x_1 \in X_1$.

On the other hand, by $h_\alpha(x) \le ||h_o||_{\alpha,X_o}^{\#} ||x_1||_{1-\alpha}$, we know that $h_\alpha(x_1) \le ||h_o||_{\alpha,X_o}^{\#}$ whenever $||x_1||_{1-\alpha} \le 1$, which means that

$||h_\alpha||_{\alpha}^{\#} = \sup\{h_\alpha(x_1): x_1 \in X_o, ||x_1||_{1-\alpha} \le 1\} \le ||h_o||_{\alpha,X_o}^{\#}$.

Thus, $||h_\alpha||_{\alpha}^{\#} = ||h_o||_{\alpha,X_o}^{\#}$.

(2) For any $\alpha \in (0,\delta)$ and $\gamma \in [0,1)$, since $\frac{h_\alpha}{X_o} = h_o$, it is obvious that

$||h_\alpha||_{\gamma}^{\#} = ||h_o||_{\gamma,X_o}^{\#}$, it follows

$N_{X_o}^{\#}(h_o,t) = \sup\{\gamma \in [0,1): ||h_o||_{\gamma,X_o}^{\#} < t\} \ge \sup\{\gamma \in [0,1): ||h_\alpha||_{\gamma}^{\#} < t\} = N^{\#}(h_\alpha,t)$.

Lemma 4.2 Let A be a semi-balanced and absorbing subset of a paratopological linear space (X_1,τ). μ_A is the minkowski functional of the set A, i.e.

$\mu_A(x_1) = \inf\{\varsigma > 0: x_1 \in \varsigma A\}\ \forall\ x_1 \in X_1$.

Put $= \{x_1: \mu_A(x_1) < 1\}$; $C = \{x_1: \mu_B(x_1) \le 1\}$

(1) $\mu_A(\varsigma x_1) = \varsigma\mu_A(x_1)\ \forall\ \varsigma > 0, \forall\ x_1 \in X_1$.

(2) If A is convex, then $\mu_A(x + y) \le \mu_A(x) + \mu_A(y), \forall\ x_1, y_1 \in X_1$.

(3) $int_\tau A \subseteq B \subseteq A \subseteq C \subseteq Cl_\tau A$

(4) The following are equivalent:

(i) $\mu_A: (X_1,\tau) \to (R,\tau(w))$ is continuous at θ,

(ii) $int_\tau A = B$,

(iii) $\theta \in int_\tau A$.

(5) If A is convex, then $\mu_A: (X_1,\tau) \to (R,\tau(w))$ is continuous at θ iff μ_A is continuous at X_1.

Theorem 4.2 Let$(X_1, N,*)$ be a fuzzy quasi-normed space and A, B two disjoint convex subsets of X with A open. Then, $\exists$ a$\delta \in (0,1]$ s.t for each $\alpha \in (0,\delta)$, there is $h_\alpha \in X^{\#}$s.t.

$h_\alpha(x_1) < h_\alpha(y_1)\ \forall\ x_1 \in A, y_1 \in B$.

Proof:

Let $\vartheta \in A, \eta \in B$ and let $\xi = \eta - \vartheta$. Since A is open and topology τ_N is translation invariant, $C = A - B + \xi$ is open. It is obvious that C is convex and $\theta \in C$.

By lemma 4.2, μ_C of C is sublinear, $\tau(w)$-continuous.

Since, $A \cap B = \phi$, then $\xi \notin C.\mu_C(\xi) \ge 1$. Let X_o be one-dimensional subspace generated by ξ. A linear functional $h_o: X_o \to R$ by $h_o(t\xi) = t\ \forall\ t \in \mathbb{R}$.

Since $h_o(t\xi) = t \le t\mu_C(\xi) = \mu_C(t\xi)$ for $t \ge 0$, and $h_o(t\xi) = t < 0 \le \mu_C(t\xi)$ for $t < 0$, it follows that

$h_o(x_1) \le \mu_C(x_1), \forall\ x_1 \in X_o$.

$\Longrightarrow\ h_o$ is $\tau(w)$-continuous.

By theorem 4.1, $\exists\ \delta \in (0,]$ s.t. $\alpha \in (0,\delta)$, there is $h_\alpha \in X^{\#}$ s.t $\frac{h_\alpha}{X_o} = h_o$.

For each $x_1 \in A$ and $y_1 \in B$, since $h(\xi) = 1$, $x - y + \xi \in C$ and C is open,

$$\Rightarrow h\phi_\alpha(x_1) - h\phi_\alpha(y_1) + 1 = \phi h_\alpha(x_1 - y_1 + \xi)$$
$$\le \mu_C(x_1 - y_1 + \xi) < 1,$$

$\Rightarrow h_\alpha(x_1) < h_\alpha(y_1)$.

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