International Journal of Theoretical & Applied Sciences, **1**(1): 128-130(2009)

A Fixed Point Theorem on Reciprocally Continuous Self Maps

Jyoti Nema and K Qureshi
Department of Mathematics
NRI Institute of Information Science and Technology Bhopal

ABSTRACT

The aim of this paper is to prove a unique common fixed point theorem which generalizes the result of Aage C.T. and Salunke J.N. by weaker conditions. The conditions continuity and completeness of a metric space are replaced by weaker conditions such as compatible pair of reciprocally continous self maps.

Keywords: Reciprocally continuous, compatible maps , fixed point, self maps

INTRODUCTION

According to G. Jungck [4] two self maps S and T of a Metric Space (X,d) are said to be compatible mappings if $\lim_{n\to\infty} d(STx_n, TSx_n)=0$ whenever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t belongs to X.

Two self maps S and T of a Metric Space (X,d) are said to be Reciprocally continuous if $\lim_{n\to\infty} STx_n$,$=St$ and $\lim_{n\to\infty} TSx_n=Tt$ whenever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t belongs to X.

Definition 1.1: A function $\varphi : [0,\infty) \to [0,\infty)$ is said to be a contractive modulus if $\varphi(0) = 0$ and $\varphi(t) < t$ for $t > 0$.

Definition 1.2: A real valued function φ defined on $X \subseteq R$ is said to be upper semicontinuous if $\lim_{n\to\infty} \varphi(t_n) \le \varphi(t)$, for every sequence $\{t_n\} \in X$ with $t_n \to t$ as $n\to\infty$.

Aage C.T. and Salunke J.N. [1] proved the following theorem:-

Theorem A: Suppose S, I, T and J are four self mappings of a complete metric space (X, d) into itself satisfying the conditions

(i) $S(X) \subset J(X)$, $T(X) \subset I(X)$.

(ii) $d^2(Sx, Ty) \le \max\{\varphi(d(Ix, Jy))\varphi(d(Ix,Sx)), \varphi(d(Ix, Jy))\varphi(d(Jy, Ty)), \varphi(d(Ix,Sx))\varphi(d(Jy, Ty)), \varphi(d(Ix,Ty))\varphi(d(Jy,Sx))\}$,

for all $x, y \in X$.

(iii) φ is contractive modulus as in definition (1.2).

(iv) one of S, I, T and J is continuous.

And if

(v) the pairs (S, I) and (T, J) are compatible of type (A).

Then S, I, T and J have a unique common fixed point.

Now we prove the following theorem.

Theorem B. Suppose S, I, T and J are four self mappings of a metric space (X, d) into itself satisfying the conditions

(i) $S(X) \subset J(X)$, $T(X) \subset I(X)$.

(ii) $d^2(Sx, Ty) \le \max\{\varphi(d(Ix, Jy))\varphi(d(Ix,Sx)), \varphi(d(Ix, Jy))\varphi(d(Jy, Ty)), \varphi(d(Ix,Sx))\varphi(d(Jy, Ty)), \varphi(d(Ix,Ty))\varphi(d(Jy,Sx))\}$,

for all $x, y \in X$.

(iii) φ is contractive modulus as in definition (1.2).

(iv) the pairs (S, I) and (T, J) are compatible pairs of reciprocally continuous mappings.

Then S, I, T and J have a unique common fixed point.

Proof: Let x_0 in X be arbitrary. Choose a point x1 in X such that Sx0 = Jx1. This can be done since $S(X) \subset J(X)$. Let x2 be a point in X such that Tx1 = Ix2. This can be done since $T(X) \subset I(X)$. In general we can choose x2n, x2n+1, x2n+2,··· such that Sx2n = Jx2n+1 and Tx2n+1 = Ix2n+2, so that we obtain a sequence

Sx0, Tx1, Sx2, Tx3, · · · . (1)

Taking condition (i). (ii) and (iii) as in Aage and Salunke [1] {Sx2n} is a Cauchy sequence and consequently the sequence (1) is a Cauchy. The sequence (1) converges to a limit z in X. Hence the subsequences {Sx2n} = {Jx2n+1} and {Tx2n−1} = {Ix2n} also converge to the limit point z.

Suppose that the pair (S,I) is compatible pair of reciprocally continuous.By the definition of reciprocally continuous, there is a sequence $<x_n>$ in X such that

Sx2n → z,Ix2n →z then SIx2n → Sz, ISx2n → Iz as n→∞. …(2)

Since the pair (S, I) is compatible we have Sx2n → z,Ix2n →z as n→∞ and

$$\lim_{n\to\infty} d(SIx_{2n}, ISx_{2n}) = 0. \quad \ldots(3)$$

using (2) and (3) we get

d(Sz, Iz) = 0 or Sz = Iz.

Since Sz=Iz, .

Now by (ii)

$$d^2(Sz, Tx2n+1) \le \max\{\varphi(d(Iz, Jx2n+1))\varphi(d(Iz,Sz)), \varphi(d(Iz, Jx2n+1))\varphi(d(Jx2n+1, Tx2n+1)), \varphi(d(Iz,Sz))\varphi(d(Jx2n+1, Tx2n+1)), \varphi(d(Iz, Tx2n+1))\varphi(d(Jx2n+1, Sz))\}.$$

Jx2n+1 → z, Tx2n+1 → z as n→∞ and Iz =Sz, so letting n→∞ we get

$$d^2(Sz, z) \le \max\{\varphi(d(Sz, z))\varphi(d(Sz, Sz)), \varphi(d(Sz, z))\varphi(d(z, z)), \varphi(d(Sz, Sz))\varphi(d(z, z)), \varphi(d(Sz, z))\varphi(d(z, Sz))\}, = \varphi(d(Sz, z))\varphi(d(z, Sz))$$

i.e. $d(Sz, z) \le \varphi(d(Sz, z)) \le d(Sz, z)$. Hence $\varphi(d(Sz, z)) = 0$ i.e. Sz = z Thus Sz = Iz = z

Further

Since $S(X) \subset J(X)$, there is a point $w \in X$ such that z = Sz = Jw.

Now we prove that Jw= Tw. Now by (ii)

$$d^2(Sz, w) \le \max\{\varphi(d(Iz, Jw))\varphi(d(Iz,Sz)), \varphi(d(Iz, Jw))\varphi(d(Jw, Tw)), \varphi(d(Iz,Sz))\varphi(d(Jw, Tw)), \varphi(d(Iz, Tw))\varphi(d(Jw, Sz))\}$$
$$= \max\{\varphi(d(Jw, Jw))\varphi(d(Jw, Jw)), \varphi(d(Jw, Jw))\varphi(d(Jw, Tw)), \varphi(d(Jw, Jw))\varphi(d(Jw, Tw)), \varphi(d(Jw, Tw))\varphi(d(Jw, Jw))\}$$

so $d^2(Jw, Tw) \le 0$ implies d(Jw, Tw) = 0 , hence z = Jw = Tw.

Since the pair (T, J) is compatible pair of reciprocally continuous.By the definition of reciprocally continuous, there is a sequence $<x_n>$ in X such that

$$Tx2n \to z, Jx2n \to z \text{ then } TJx2n \to Tz,\ JTx2n \to Jz \text{ as } n\to\infty. \quad \ldots(4)$$

Since the pair (T,J) is compatible we have $Tx2n \to z, Jx2n \to z$ as $n\to\infty$ and

$$\lim_{n\to\infty} d(TJx_{2n}.,JTx_{2n}) = 0. \quad \ldots(5)$$

using (4) and (5) we get

$d(Tz, Jz) = 0$, hence $Tz = Jz$. Now

$$\begin{aligned} d^2(z, Tz) &= d^2(Sz, Tz) \\ &\leq \max\{\varphi(d(Iz, Jz))\varphi(d(Iz,Sz)), \varphi(d(Iz, Jz))\varphi(d(Jz,Tz)), \\ &\quad \varphi(d(Iz,Sz))\varphi(d(Jz,Tz)), \varphi(d(Iz, Tz))\varphi(d(Jz,Sz))\} \\ &= \max\{\varphi(d(z, Tz))\varphi(d(z, z)), \varphi(d(z, Tz))\varphi(d(Tz,Tz)), \\ &\quad \varphi(d(z, z))\varphi(d(Tz,Tz)), \varphi(d(z, Tz))\varphi(d(Tz, z))\} \\ &= \varphi(d(Tz, z))\varphi(d(Tz, z)) \end{aligned}$$

implies that $d(Tz, z) \leq \varphi(d(Tz, z)) \leq d(Tz, z)$.
Hence $\varphi(d(Tz, z)) = 0$ i.e. $Tz = z$ and $z = Tz = Jz$. So z is a common fixed point of S, I, J and T.

Uniqueness:
Let z' be another common fixed point of S, I, J and T. i.e. $z' = Sz' = Iz' = Tz' = Jz'$. From condition (ii) we have

$$\begin{aligned} d^2(z,z') &= d^2(Sz, Tz') \\ &\leq \max\{\varphi(d(Iz, Jz'))\varphi(d(Iz,Sz)), \varphi(d(Iz, Jz'))\varphi(d(Jz', Tz')) \\ &\quad \varphi(d(Iz,Sz))\varphi(d(Jz', Tz')), \varphi(d(Iz, Tz'))\varphi(d(Jz',Sz))\} \\ &= \max\{\varphi(d(z,z')\varphi(d(z, z)), \varphi(d(z,z'))\varphi(d(z',z')), \\ &\quad \varphi(d(z, z))\varphi(d(z',z')), \varphi(d(z,z'))\varphi(d(z', z))\} \end{aligned}$$

Therefore $d(z,z') \leq \varphi(d(z,z')) \leq d(z,z')$ i.e. $\varphi(d(z,z')) = d(z,z')$. Thus $d(z,z') = 0$ i.e. $z' = z$. Hence the common fixed point is unique.

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