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Fixed point theorem of a certain class of mapping in p-uniform convex banach space

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Abstract : In this paper a fixed point theorem is proved in p-Uniformly Banach space for a class of mappings S and T satisfying the condition $\|T^{n+1}x - T^n y\| \leq a\| S^{n+1}x - x\| + b\| S^n y - y\|$. For all x and y in domain and n = 1, 2, 3,………Further some fixed point theorems for such mappings will proved in Hilbert space and L^p space.

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INTRODUCTION

Let K be a nonempty subset of a Banach space X. A mapping T:K→K is said to be uniformly α Lipschitzian if $\| T^n x - T^n y \| \leq \alpha \| x - y \|$||| for all x and y in K and $n \geq 1$. This class of mappings have been studied by many authors Goebel and Kirk [2] proved that such T has a fixed point if K is is a bounded convex subset of uniformly convex Banach space X and $\alpha < M$, M being the unique solution the equation.

$M.(1 - \delta_X(1/M)) = 1$ and $\delta_X(.)$ is the modulus of convexity of X. For Hilbert space H, $M = (\sqrt{5})/2$ and for L^p space $M = (1 + p/2)^{1/p}$. *Xu* [5] extend the result of Lifshitz [4] and Lim [3] which extend the result of Geobel and Kirk [2].

In this paper these results for class of intimate mappings whose nth iterate is

$$\| T^{n+1}x - T^n y \| \leq \alpha \| S^{n+1}x - x \| + \beta \| S^n y - y\| \quad ...(1)$$

Preliminaries

The normal structure coefficient $N(X)$ of X is defined by Bynum[1].

K is a bounded convex subset of X consisting of more than one point, then

$$N(X) = \inf \; [\text{diam } K / \gamma_K(K)]$$

where $\gamma_K(K) = \inf_{x \in K} \; (\sup_{x \in K} \| x - y \|)$ is the Chebyshev radius of K relative to itself.

X is said to have uniformly normal structure if $N(X) > 1$ and for Hilbert space.

H the $N(H) = \sqrt{2}$ and $N(Lp)$ min $\{2^{1/p}, 2^{(p-1)/p}\}$ for $1 < p < \infty$.

Let $p > 1$ and for λ in [0,1] and $W_p(p)$ be the function $\lambda(1 - \lambda)^p + \lambda^p(1 - \lambda)$.

The function $\lambda(1 - \lambda)^p + \lambda^p(1 - \lambda)$. The functional $\|.\|^p$ is said to be uniformly convex on Banach space X if there exists $c > 0$ such that for all $\lambda \in [0, 1]$ and for all x, y in X the following inequality holds:

$$\| \lambda x + (1 - \lambda)y \|^p \leq \lambda \|x\|^p + (1 - \lambda) \|y\|^p - W_p(\lambda).c_p \|x - y\|^p \quad ...(2)$$

Xu [5] proved that the functional $\|.\|^p$ is said to be uniformly convex on whol Banach space X if and only if X is p-uniformly convex *i.e.*, there exists a constant $c > 0$ such that moduli of convexity, $\delta_X(\varepsilon) \geq c.\varepsilon^p$ for all $0 \leq \varepsilon \leq 2$.

Before presenting the main result we need the following lemma of [5] :

Lemma 1 : Let $p > 1$ and X be the p-uniformly convex Banach space and K be a nonempty closed convex subset of X and $\{x_n\}$ in X be a bounded sequence. Then there exists a unique z in K such that

$$\limsup_{n \to \infty} \| x_n - z \|^p \leq \limsup_{n \to \infty} \| x_n - x \|^p - c_p. \| x - z \|^p \quad ...(3)$$

for every x in K , where is c_p constant as in (2).

MAIN RESULT

Theorem : Let $p > 1$ and X be the p-uniformly convex Banach space and K be a nonempty closed convex subset of X and S, $T : K \to K$ be mappings whose n^{th} iteration satisfy the inequality (1) with

$$[(\alpha + \beta)^p.\{(\alpha + \beta)^p - 1\}/c_p.N^p]^{1/p}$$

where N be the normal structure coefficient of X and is c_p constant as in (2). Suppose there is an x_0 in K for which $\{S^n x_0\}$ is bounded, then S and T have a fixed point in K.

Proof : Since $\{T^n x_0\}$ and $\{S^n x_0\}$ are bounded and so $\{T^n x\}$ and $\{S^n x\}$ are bounded for all x in K, by lemma 1 we can inductively construct a $\{x_n\}$ in K as follows for each $m \geq 0$, x^{m+1} be the asymptotic centre of sequences $\{T^{n+1} x_m\}$ and $\{S^{n+1} x_m\}$ in K. Let

$$\gamma_m = \limsup_{n\to\infty} \| T^{n+1} x_m - x_{m+1} \|$$

and
$$D_m = \limsup_{n\to\infty} \| x_m - S^n x_m \|$$

Then

$$\| T^{i+1}x - T^{j+1}y \| \leq \| S^{i+1}x - S^{j+1}y \|$$
$$\leq 1\ \alpha \| S^i x - x \| + \beta \| S^j y - y \| \qquad ...(4)$$

Then by the result of Lim [3] and (1), we have

$$\gamma_m = \lim_{n\to\infty} \sup \| T^{n+1} x_m - x_{m+1} \|$$

$$\leq 1/N. \lim_{n\to\infty} \sup\{\| T^{n+1} x_m - T^{j+1} x_{m+1} \| : i, j \geq k\}$$

$$\gamma_m = [(\alpha + \beta)/N] .D_m \qquad ...(5)$$

For each $m \geq 1$ and all $n > r$, $s \geq 1$, we have

$$\| \lambda x_{m+1} + (1-\lambda) T^{r+1} x_{m+1} - T^{s+1} x_m \|^p + W_p(\lambda).c_p \| x_{m+1} - T^{r+1} x_m \|^p$$
$$\leq \| \lambda x_{m+1} + (1-\lambda) S^r x_{m+1} - S^s x_m \|^p$$
$$+ W_p(\lambda).c_p \| x_{m+1} - S^r x_m \|^p$$
$$\leq \lambda \| x_{m+1} - S^n x_m \|^p + (1-\lambda) \| S^r x_{m+1} - S^n x_m \|^p$$
$$\leq \lambda \| x_{m+1} - S^n x_m \|^p + (1-\lambda)$$
$$\{\alpha \| x_{m+1} - S^{n-s} x_m \|^p + \beta \| S^n x_m - x_{m+1} \|^p\}$$

Now taking limit superior, we get

$$\gamma_m^p + W_p(\lambda).c_p \| x_{m+1} - T^{r+1} x_{m+1} \|^p$$
$$\leq \{\lambda + (1-\lambda)(\alpha + \beta)^p\}.\gamma_m^p$$
$$[(1-\lambda).\{(\alpha + \beta)^p - 1\}/c_p W_p(\lambda)].\gamma^p_m$$
$$\leq [(1-\lambda).\{(\alpha + \beta)^p - 1\}/c_p W_p(\lambda)]\,[(\alpha + \beta)^p / N^p] D^p_m$$

On taking limit $\lambda \to 1$,we get

$$D_{m+1} = [(\alpha + \beta)^p.\{(\alpha + \beta)^p - 1\}/c_p.N^p]^{1/p} D_m$$
$$D_{m+1} = A \,.\, D_m$$

where $[(\alpha + \beta)^p.\{(\alpha + \beta)^p - 1\}/c_p.N^p]^{1/p} < 1$

$$\Rightarrow D_{m+1} = A \,.\, D_m \leq A^2.D_{m-1}$$
$$\leq A^3 \,.\, D_{m-2} \ldots \leq A^m \,.\, D_1.$$

As $m \to \infty$, $\| x_{m+1} - x_m \| \to 0$, it follows that the sequence $\{x_n\}$ is a Cauchy sequence. Let $z = \lim_{n\to\infty} x_m$, then from triangle inequality and (4)

$$\| z - Tz \| \leq \| z - x_m \| + \| x_m - Tx_m \| + \| Tx_m - Tz \|$$
$$\leq \| z - x_m \| + \| x_m - Tx_m \| + \alpha \| Sx_m - z \| + \beta \| Sz - x_m \|$$

And hence $Tz = z$ and also $Sz = z$.

Corollary 1 : Let K be a nonempty closed convex subset of Hilbert space H and S, $T : K \to K$ be mappings whose n^{th} iteration satisfy the inequality (1) with

$$[(\alpha + \beta)^2.\{(\alpha + \beta)^2 - 1\}/c_p \,.\, 2^{(p-1)/p}]^{1/2} < 1$$

where $1 < p \leq 2$, N be the normal structure coefficient of X and is c_p constant as in (2). Suppose there is an x_0 in K for which $\{S^n x_0\}$ is bounded, then S and T have a fixed point in K.

Corollary 2 : Let K be a nonempty closed convex subset of (L^p) space H and S, $T : K \to K$ be mappings whose n^{th} iteration satisfy the inequality (1) with

$$[(\alpha + \beta)^2.\{(\alpha + \beta)^2 - 1\}/c_p \,.\, 2^{(p-1)/p}]^{1/2} < 1 \text{ for } 1 < p \leq 2$$

and
$$[(\alpha + \beta)^2.\{(\alpha + \beta)^2 - 1\}/c_p \,.\, 2]^{1/2} < 1 \text{ for } 2 < p \leq \infty$$

where N be the normal structure coefficient of X and is c_p constant as in (2). Suppose there is an x_0 in K for which $\{S^n x_0\}$ is bounded, then S and T have a fixed point in K.

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