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Integral Transform Technique to Remove Singularity and Solution of an Unsteady-state Mass Transport Equation for Chlorine Concentration Decay in a Drinking Water Pipe Line System

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ABSTRACT: The mass transport equation for chlorine concentration decay in drinking water pipe line system by considering advection term in axial direction, first order decay kinetics and transport of chlorine from bulk flow to the wall, is solved using integral transform technique. For small Peclet number P_e the value mass chlorine concentration is obtained at different locations and shown graphically. Hankel transformation can efficiently be used to remove singularity in such type of transport equation.

Key words: Mass transport, Bulk flow, Advection, Peclet number, Chlorine concentration

INTRODUCTION

There are many models available in the literature for chlorine concentration decay in water pipe line network but every model has its own limits. Clark *et al.* [4] considered how chlorine residuals can vary throughout the day at different locations in the distributive systems. Clark *et al.* [5] used first order kinetics and rate of chlorine decay in their model. They showed that pipe radius and the fluid velocity affect the propagation of chlorine residual levels, disinfection efficiency and the formation of disinfection by-products. Biswass *et al.* [1] considered two dimensional axi-symmetric steady state model without considering the transport of chlorine from bulk flow to the pipe wall Reddy *et al.* [14] studied the weighted least-square method for some parameter estimation in water distribution network, like model pressure heads, pipe flow, head loss in pipes and consumptions in flows. David and Bryan [6] considered an adjective transport model by neglecting the contribution of radials as well as axial diffusion terms. A comparison was made between the formulation and computational performance of numerical models for modeling the transient behavior of water quality in drinking water distribution system by Rossman and Paul [15]. Sam *et al.* [16] considered a model for the study of the behavior of bacterial biomass distribution networks taking into account biodegradable dissolved organic carbon, temperature, residual chlorine and PH etc. Kumar *et al.* [10] studied the biological pharmacokinetic effects on human health of some water soluble chemicals. Rossman and Paul [15] considered one dimensional mass conservation equation for a dilute concentration of total free chlorine in water flowing through a cross section of a pipe which gives a mechanism for considering the loss or growth of a substance by reaction as it travels through the water distribution system. Rashidual Islam *et al.* [13] used inverse method to evaluate the source method to calculate the source concentration of chlorine which is required to meet a specified value at a particular point in the network. They claimed that their model directly calculate the chlorine concentration needed at source. In their model they have considered one dimensional transport equation with first order reaction but they have not considered diffusion term in their model. Clark [3] developed a mathematical model to predict the total trihalomethens and chlorine residuals based on the consumption on the chlorine. He suggested that the model can be used to evaluate the balance between microbiological and disinfection by-products resist associated with chlorine that was used to disinfect the drinking water. He has not acknowledged bromide and brominated by-products in his model. Hoefel *et al.* [8] in micro trial resistant to chlorination has observed both of these in lab studies and in full scale chlorine disinfection practice for water and waste-water treatment. Osman and Metin [12] studied a two dimensional mathematical model for chlorine concentration decay in water and solved the chlorine transport equation numerically by finite difference method. They have not acknowledged transport of chlorine from bulk flow towards the pipe wall. James *et al.* [9] distinguished six different kinetic models for the decay of free chlorine by taking different samples from different water reservoirs as well as different source. They advanced that the first order kinetics decay models are reasonable in general. Velitchko *et al.* [17] acknowledged one dimensional mass transport equation and studied an Eulerian and Lagrangian numerical solution for that.

They suggested that the model could be applied to simulate fluoride and chlorine propagation in a real network for which published data of field measurements. General regression neural networks are discussed for forecasting chlorine residuals in South Australia by Bowden Gavin *et al*. [2]. They claimed that the general regression neural networks models were available to predict up to a very good label of accuracy of chlorine labels in drinking water and up to 72 hours in advance. Gibbs *et al.* [7] used three different data driven techniques to predict chlorine concentrations in the Hope valley water distribution system of South Australia. They claimed that data driven techniques were relatively successful in predicting chlorine concentration in the water distribution system. The scholars suggested that their model gave a more accurate approximate solution of the two dimensional steady-state chlorine transport equation under turbulent condition. The main drawback of model is that they have not considered chlorine decay to be time dependent. Kumar and Mishra [11] studied the effect of transport ratio on source term by considering the one dimensional mass transport equation by ignoring advection term. In this paper I have considered two dimensional and unsteady state mathematical model for chlorine dispersion for axi-symmetric flow in a pipe line system. The transport of chlorine in axial direction due to advection term, first order decay kinetics and transport of chlorine from bulk flow to the wall are also considered that fulfill almost all requirement of a complete model for chlorine concentration decay in drinking water pipe line system.

MATHEMATICAL MODEL

Consider the mass transport equation for chlorine concentration as

$$\frac{\partial C}{\partial t}-D_x\frac{\partial^2 C}{\partial x^2}+U\frac{\partial C}{\partial x}-D_r(\frac{\partial^2 C}{\partial r^2}+\frac{1}{r}\frac{\partial C}{\partial r})=-k_bC-\frac{k_f(C-C_w)}{r_h} \quad (1)$$

where D_x is diffusion coefficient in x direction, D_r is diffusion coefficient in r direction , U is initial velocity component along x axes, k_b and k_f are the chlorine decay rate constant for bulk flow(s^{-1}) and mass transfer coefficient (m/s) respectively C_w is the chlorine concentration at wall (kg/m^3) and r_h is the hydraulic radius of the pipe wall. As we see the term $\frac{1}{r}\frac{\partial C}{\partial r}$ is present in the above mass transport equation so there is a singularity at $r=0$ i.e. on the central line of pipe. To overcome the problem of singularity we have used finite Hankle Transformation technique, Senddon [18]. Assuming that the reaction of chlorine at the pipe wall is of first order with respect to the wall concentration C_w and that it proceeds at the same rate as chlorine is transported to the wall gives the following mass balance equation for the chlorine at the wall.

$$k_f(C-C_w)=k_wC_w \quad (2)$$

Substituting the value of C_w from equation (2) into equation (1)

$$\frac{\partial C}{\partial t}-D_x\frac{\partial^2 C}{\partial x^2}+U\frac{\partial C}{\partial x}-D_r(\frac{\partial^2 C}{\partial r^2}+\frac{1}{r}\frac{\partial C}{\partial r})+k_bC+\frac{k_fk_wC}{r_h(k_f+k_w)}=0 \quad (3)$$

The initial and boundary conditions are

$$C(x,r,0)=0 \quad (4) \qquad C(0,r,t)=c_0 \quad (5)$$

$$\frac{\partial C}{\partial x}=0 \quad as\ x\to\infty, t\geq 0 \quad (6)$$

and wall condition is $\frac{\partial C}{\partial r}=0$ at $r=r_0$ (7)

Introducing the following non dimensional quantities are defined by

$$C'=\frac{C}{C_0},\ x'=\frac{x}{L},\ r'=\frac{r}{r_0},\ t'=\frac{Ut}{L} \quad (8)$$

where r_0 is the pipe radius.

$$\frac{\partial C}{\partial t}-\frac{1}{p_e}\frac{\partial^2 C}{\partial x^2}+\frac{\partial C}{\partial x}-\frac{D_r}{D_x}\frac{L^2}{r_0^2}\frac{1}{p_e}(\frac{\partial^2 C}{\partial r^2}+\frac{1}{r}\frac{\partial C}{\partial r})+KC=0 \quad (9)$$

where $K=k_b+\dfrac{k_w k_f}{r_h\left(k_w+k_f\right)}$ and $p_e=\dfrac{UL}{D_x}$

The initial and boundary condition are

$$C(x,r,t)=0,\ t=0 \quad (10)$$

$$C(x,r,t)=1,\ x=0$$

(11) $\dfrac{\partial C}{\partial r}=0,\ r=1$ …. (12)

$$\frac{\partial C}{\partial x}=0,\ t\geq 0,\ \ x\rightarrow\infty \quad (13)$$

Applying finite Hankel transformation on equation (9) and equations (10) to (13), gives

$$\frac{\partial C_H}{\partial t}-\frac{1}{p_e}\frac{\partial^2 C_H}{\partial x^2}+\frac{\partial C_H}{\partial x}+\lambda C_H=0 \quad (14)$$

where $\lambda=\dfrac{D_r L^2\lambda_n^2}{D_x r_0^2 p_e}+K$

λ_n is finite Hankel transformation parameter as determined by the transcendental equation $\dfrac{d(J_0(\lambda_n))}{dr}=0$. $J_0(\lambda_n)$ is the zero order Bessel function of the first kind.

$C_H(x,\lambda_n,t)$ is the second kind finite Hankel transformation of $C(x,r,t)$ as defined by the following conjugate equation

$$C_H(x,\lambda_n,t)=\int_0^1 rC(x,r,t)J_0(\lambda_n r)dr \quad (15)$$

$$C(x,r,t)=2\sum_{n=0}^{\infty}C_H(x,\lambda_n,t)\frac{J_0(\lambda_n r)}{\left|J_0(\lambda_n)\right|^2} \quad (16)$$

The initial and boundary conditions become

$C_H(x,\lambda_n,0)=0$ …………..(17) $\quad C_H(0,\lambda_n,t)=\dfrac{J_1(\lambda_n)}{\lambda_n}\ when\ \lambda_n\neq 0$ (18)

$$\frac{\partial C_H}{\partial x}=0,\quad as\ x\rightarrow\infty\ \ ,t\geq 0 \quad (19)$$

Again introducing following transformation to solve equation (14)

$$C_H(x,\lambda_n,t)=P(x,t)\exp\left[\frac{p_e x}{2}-\left(\frac{p_e}{4}+\lambda\right)t\right] \quad (20)$$

Equation (14) reduced into

$$\frac{\partial P}{\partial t}-\frac{1}{p_e}\frac{\partial^2 P}{\partial X^2}=0 \quad (21)$$

The initial and boundary condition (17) to (19) become

$$P\left(x,t\right)=0,\quad x\geq 0,\quad t=0 \tag{22}$$

$$P(x,t)=\frac{J_1(\lambda_n)}{\lambda_n}\exp\left[\left(\frac{p_e}{4}+\lambda\right)t\right],\ x=0,\ \ t>0\ \ when\lambda_n\neq 0 \tag{4.23}$$

$$\frac{dP}{dx}=0,t\geq 0,x\rightarrow\infty \tag{24}$$

Solving equation (21) together with initial boundary conditions (22) to (24) by Laplace Transformation technique and then putting the value of

$P(x,t)$ in equation (20), we get

$$\begin{aligned}C_H\left(x,\lambda_n,t\right)=&\frac{1}{2}\frac{J_1(\lambda_n)}{\lambda_n}\left[\exp\left(\frac{p_e+(p_e^2+4\lambda p_e)^{\frac{1}{2}}}{2}\right)x.erfc\left(\frac{\sqrt{p_e}x+\sqrt{p_e+4\lambda t}}{2\sqrt{t}}\right)\right]\\&+\frac{1}{2}\frac{J_1(\lambda_n)}{\lambda_n}\left[\exp\left(\frac{p_e-(p_e^2+4\lambda p_e)^{\frac{1}{2}}}{2}\right)x.erfc\left(\frac{\sqrt{p_e}x-\sqrt{p_e+4\lambda t}}{2\sqrt{t}}\right)\right]\end{aligned} \tag{25}$$

where $\lambda_n\neq 0$

Finally putting equations (25) in equation (16), we get

$$\begin{aligned}C(x,r,t)=&\sum_{n=0}^{\infty}\left[\exp\left(\frac{p_e+(p_e^2+4\lambda p_e)^{\frac{1}{2}}}{2}\right)x.erfc\left(\frac{\sqrt{p_e}x+\sqrt{p_e+4\lambda t}}{2\sqrt{t}}\right)\right]\frac{J_0(\lambda_n r)}{\left|J_0(\lambda_n)\right|^2}\frac{J_1(\lambda_n)}{\lambda_n}\\&+\sum_{n=0}^{\infty}\left[\exp\left(\frac{p_e-(p_e^2+4\lambda p_e)^{\frac{1}{2}}}{2}\right)x.erfc\left(\frac{\sqrt{p_e}x-\sqrt{p_e+4\lambda t}}{2\sqrt{t}}\right)\right]\frac{J_0(\lambda_n r)}{\left|J_0(\lambda_n)\right|^2}\frac{J_1(\lambda_n)}{\lambda_n}\end{aligned} \tag{26}$$

where $\lambda_n\neq 0$

RESULTS AND DISCUSSION

We have not considered the case for large Peclet number P_e as the equation reduces to a simple linear equation. For small Peclet number P_e the solution of equation (9) is given by equation (20). Initially chlorine is supposed to injected at x = 0, r = 0. To observe the effect of diffusivity, fluid velocity and chlorine consumption rate on the chlorine concentration in the water figure 1 to figure 3 are plotted for various values of parameters. It is clear from Fig. 1 that chlorine concentration decreases very fast along the axial distance while slowly along radial distance. Chlorine concentration decreases rapidly from x = 0 to x = 0.6 and after that it becomes constant. It appears that after x = 0.6 concentration is zero. In fact it is not zero and x = 0.6, C = 0.007125 while at x = 0.8, C = 0.000267, and x = 1.0, C = 0.00000391 (at r = 0) which are very near to zero. To maintain the safe limit for chlorine concentration we have to inject chlorine again after x = 0.4 and before x = 0.6. The variation of chlorine concentration along radial direction is very small and it is difficult to observe from the figure. But we can see the change from the numerical value as at x = 0.1, r = 0, C = 0.643566 while at x = 0.1, $r = 1$ (i.e. at the wall of pipe) C = 0.469745. The effect of chlorine consumption rate K (which depends upon transport of chlorine from bulk flow to the wall, chlorine decay rate constant and mass transfer coefficient) can be observed by comparing figure 1 (for K = 0.001) and Fig. 2 (for K = 0.01).

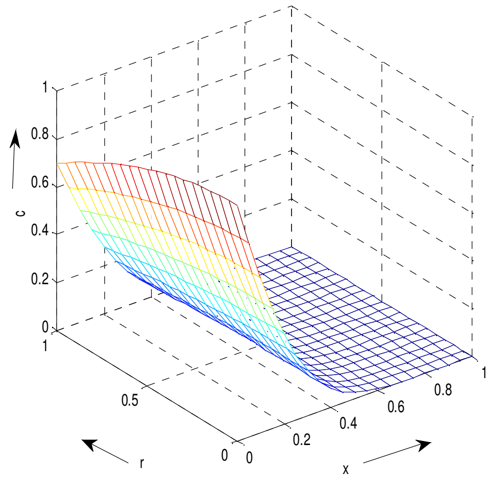

Fig. 1. Change in chlorine concentration with x and r, K=0.001.

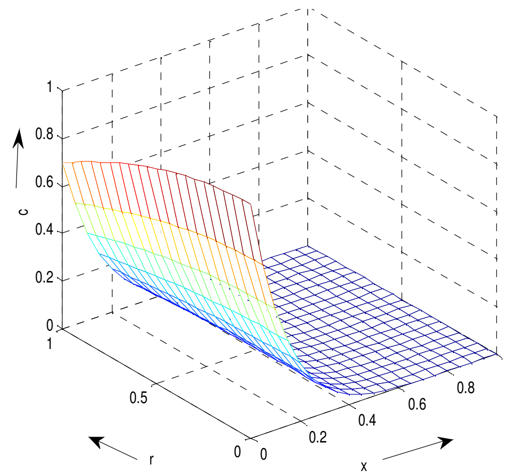

Fig. 2. Change in chlorine concentration with x and r, K=0.01.

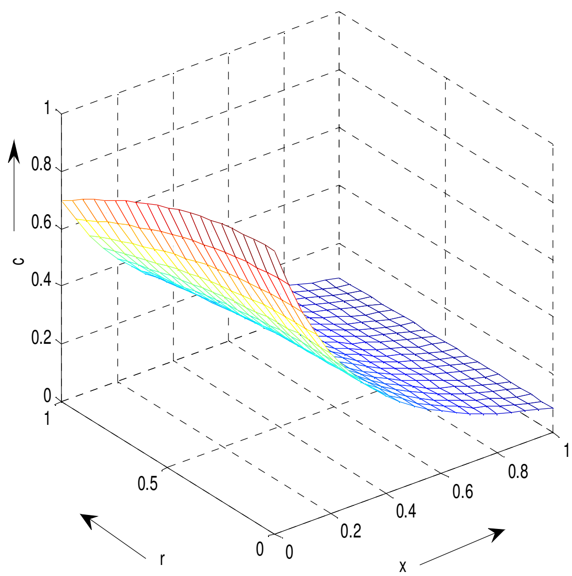

Fig. 3. Variation of chlorine concentration with x and r, $K = 0.001$, $t = 0.8$.

As the chlorine consumption rate K increases from $K = 0.001$ to $K = 0.01$, the more chlorine is transported towards the pipe wall and less chlorine remains in the bulk flow. For $K = 0.001$ (Fig. 1) the chlorine concentration approaches to zero after $x = 0.6$ while for $K = 0.01$ (Fig. 2) chlorine concentration approaches to zero before $x = 0.6$. As this model is time dependent so to see the dependence of chlorine concentration on time we compare Fig. 1 and Fig. 3. At $t = 0.1$, $C = 0.00000391$ (in fig.1 at $x = 0$, $r = 0$) while at $t = 0.8$, $C = 0.0857123$ (in fig. 3 at $x = 1$, $r = 0$) which is true fact since initially the chlorine injected at $x = 0$ and $r = 0$ and it takes some time to reach at $x = 1.0$ (i.e. end of the pipe).

CONCLUSION

A two dimensional unsteady state mathematical model for transport of chlorine in an axi-symmetric pipe is considered in this paper together with source term and first order decay kinetics which fulfill almost all the requirements of a complete model.

The analytical solution for chlorine concentration in the above model is obtained by using Hankel transformation and the graphical results for low Peclet number are shown. The model can be used effectively to locate the position for buster chlorination to maintain the safe limit of the drinking water.

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