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Common Fixed Point Theorems in Fuzzy Normed Spaces

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ABSTRACT : In this paper, we prove a common fixed point theorem for four self-maps in fuzzy normed space using the concept of compatibility, which generalizes the result of Singh *et. al.* [5].

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I. INTRODUCTION AND PRELIMINARY CONCEPTS

Zadeh [6] introduced the concept of fuzzy sets in 1965. Many researches have been done using this concept in different spaces. In 1999, Jose and Santiago [2] introduced the concept of Fuzzy norm on a real or complex vector space and defined Fuzzy normed space (called F-normed space) by modifying the definition of F-normed spaces given by George [1] in 1995. Jungck [3] introduced the concept of compatible mappings for a pair of self-maps. The concept of compatibility in fuzzy metric space was introduced by Mishra *et. al.* [4].

Definition 1. [2] A 3-tuple, $(X, N*)$ is said to be a F-normed space if X is a real or complex vector space, * is a continuous t-norm and N is function on $X \times (0, \infty)$ satisfying the following conditions :

(5.1.1) $N(x, t) > 0$;

(5.1.2) $N(x, t) = 1$ if and only if $x = 0$;

(5.1.3) $N(kx, t) = N(x, t/|k|)$;

(5.1.4) $N(x, t) * N(y, s) \leq N(x + y, t + s)$;

(5.1.5) $N(x, .) : (0. \infty) \rightarrow [0, 1]$ is continuous,

for all $x, y \in$ X and $t, s > 0$.

Remark 1. [2] Let $(X, N, *)$ be a F-normed space. For $x, y \in X$, $t > 0$, define $M(x, y, t) = N(x - y, t)$. Then $(X, M, *)$ is a fuzzy metric space.

Definition 2. [2] A sequence $\{x_n\}$ in a F-normed space $(X, N, *)$ is said to be convergent to an element $x \in X$ if and only if given $t > 0$, $0 < r < 1$, there exists an $n_0 \in J$ such that

$$N(x_n - x, t) > 1 - r, \text{ for every } n \geq n_0.$$

Definition 3. [2] A sequence $\{x_n\}$ in a F-normed space $(X, N, *)$ is said to be *F*-Cauchy sequence if and only if for every ε such that $0 < \varepsilon < 1$, $t > 0$, there exists an $n_0 \in J$ such that

$$N(x_n - x_m, t) > 1 - \varepsilon, \text{ for every } n, m \geq n_0.$$

Definition 4. [2] A F-normed space $(X, N, *)$ is said to be complete if every F-Cauchy sequence in X converges to an element in X.

Lemma 1. [5] A sequence $\{y_n\}$ in a F-normed space $(X, N, *)$ is F-Cauchy if there exists a constant $k \in (0, 1)$ such that

$$N(y_n - y_{n+1}, kt) \geq N(y_{n-1} - y_n, t) \text{ for all } n \in N, t > 0.$$

Definition 5. Let A and B be self-mappings in a F-normed space $(X, N, *)$. The pair (A, B) is said to be compatible if

$$\lim_{n\to\infty} N(ABx_n - BAx_n, t) = 1 \text{ for all } t > 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x, \text{ for some } x \in \text{X}.$$

II. MAIN RESULT

Singh *et. al.* [5] established a result regarding fixed points in fuzzy normed space, which is as follows :

Theorem 1. [5] Let f and g be self-maps of a complete F-normed space $(X, N, \min)$ such that, for $k \in (0, 1)$

$N(fu - gv, kt) \geq \min\{N(u - fu, t), N(v - gv, t), N(u - gv, 2t), N(v - fu, t)\}$, holds for all u, v in X, $t > 0$,

$$f(X) \subseteq g(X).$$

Then f and g have a unique common fixed point.

In this paper, a common fixed point theorem for four self-mappings in F-normed space is proved which generalizes the result of Singh *et. al.* [5] as our result is proved for four self-mappings by using compatibility and different functional inequality.

Theorem 2. Let A, B, S and T be self-maps of a complete F-normed space $(X, N, \min)$ satisfying the following conditions:

(2.2.1) for all x, y in X, $k \in (0, 1)$, $t > 0$

$N(Ax - By, kt) \geq \min\{N(Sx - Ty, t), N(Ax - Sx, t), N(By - Ty, t), N(Ax - Ty, t), N(Sx - By, 2t)\};$

(2.2.2) $A(X) \subseteq T(X), B(X) \subseteq S(X);$

(2.2.3) one of A, B, S or T is continuous;

(2.2.4) the pairs $[A, S]$ and $[B, T]$ are compatible;

Then A, B, S and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. As $A(X) \subseteq T(X)$ and $B(X) \subseteq S$(X) then there exists $x_1, x_2 \in X$ such that $Ax_0 = Tx_1, Bx_1 = Sx_2$. Construct a sequence $\{y_n\}$ in X such that (2.2.5) $y_{2n+1} = Tx_{2n+1} = Ax_{2n}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$ for $n = 1, 2, 3, \ldots$.

Now to prove $\{y_n\}$ is a Cauchy sequence, we shall prove that (2.2.6) $N(y_{2n+1} - y_{2n+2}, kt) \geq N(y_{2n} - y_{2n+1}, t) \;\forall\; t > 0$

Suppose this is not true, then we get

(2.2.7) $N(y_{2n+1} - y_{2n+2}, kt) < N(y_{2n} - y_{2n+1}, t) \;\forall\; t > 0.$

From (2.2.1) and (2.2.5), we have

$N(y_{2n+1} - y_{2n+2}, kt)$

$= N(Ax_{2n} - Bx_{2n+1}, kt)$

$\geq \min\{N(Sx_{2n} - Tx_{2n+1}, t), N(Ax_{2n} - Sx_{2n}, t), N(Bx_{2n+1} - Tx_{2n+1}, t), N(Ax_{2n} - Tx_{2n+1}, t), N(Sx_{2n} - Bx_{2n+1}, 2t)\}.$

$= \min\{N(y_{2n} - y_{2n+1}, t), N(y_{2n+1} - y_{2n}, t), N(y_{2n+2} - y_{2n+1}, t), N(y_{2n+1} - y_{2n+1}, t), N(y_{2n} - y_{2n+2}, 2t)\}.$

$= \min\{N(y_{2n} - y_{2n+1}, t), N(y_{2n+2} - y_{2n+1}, t), 1\; N(y_{2n} - y_{2n+1}, t), N(y_{2n+1} - y_{2n+2}, t).$

$(N(y_{2n+1} - y_{2n+2}, kt) \geq \min\{N(y_{2n} - y_{2n+1}, t), N(y_{2n+2} - y_{2n+1}, t), 1\}.$

Using (2.2.7), we get)

$N(y_{2n+1} - y_{2n+2}, kt) > \min\{N(y_{2n+1} - y_{2n+2}, kt), N(y_{2n+1} - y_{2n+2}, t)\}.$

which is a contradiction. Hence (2.6) is true.

In general

$N(y_n - y_{n+1}, kt) \geq N(y_{n-1} - y_n, t) \;\forall\; t > 0, \;\forall\; n \in N.$

Therefore, by lemma 1.1, $\{y_n\}$ is a Cauchy sequence and by completeness of F-normed space, it converges to some point z in X. Thus the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n-1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ of sequence $\{y_n\}$ also converges to z in X.

Suppose S is continuous and the pair (A, S) is compatible, we have

$SAx_{2n} \to Sz, S^2x_{2n} \to Sz$ and $ASx_{2n} \to Sz.$

Step 1. Putting $x = Sx_{2n}$ and $y = x_{2n-1}$ in (2.2.1), we have

$N(ASx_{2n} - Bx_{2n-1}, kt) \geq \min\{N(SSx_{2n} - Tx_{2n-1}, t),$

$N(ASx_{2n} - SSx_{2n}, t), N(Bx_{2n-1} - Tx_{2n-1}, t), N$

$(ASx_{2n-1} - Tx_{2n-1}, t), N(SSx_{2n} - Bx_{2n-1}\; 2t)\}.$

Letting $n \to \infty$ and using above results, we get

$N(Sz - z, kt) \geq \min\{N(Sz - z, t), 1\}.$

$N(Sz - z, kt) \geq 1$

which implies that

$Sz = z.$

Step 2. Putting $x = z$ and $y = x_{2n-1}$ in (2.2.1), we have $N(Az - Bx_{2n-1}, kt) \geq \min\{N(Sz - Tx_{2n-1}, t), N(Az - Sz, t), N(Bx_{2n-1} - Tx_{2n-1}, t), N(Az - Tx_{2n-1}, t), N(Sz - Bx_{2n-1}, 2t)\}.$

Letting $n \to \infty$ and using above results, we get

$N(Az - z, kt) \geq \min\{N(Az - z, t), 1\}.$

$N(Az - z, kt) \geq 1$

which implies that $Az = z$.

Step 3. Since $A(X) \subseteq T$(X), there exists $u \in X$ such that $z = Az = Tu$. Putting $x = z$ and $y = u$ in (2.2.1), we have $N(Az - Bu, kt) \geq \min\{N(Sz - Tu, t), N(Az - Sz, t), N(Bu - Tu, t), N(Az - Tu, t), N(Sz - Bu, 2t)\}.$

Using above results, we get

$N(z - Bu, kt) \geq \min\{1, N(z - Bu, t)\}.$

$N(z - Bu, kt) \geq 1$

which implies that $z = Bu$.

Since B ant T are compatible and $Bu = Tu$ implies

$N(BTu - TBu, t) = 1.$

Therefore

$Bz = BTu = TBu = Tz.$

Step 4. Putting $x = z$ and $y = z$ in (2.2.1), we have $N(Az - Bz, kt) \geq \min\{N(Sz - Tz, t), N(Az - Sz, t), N(Bz - Tz, t), N(Az - Tz, t), N(Sz - Bz, 2t)\}.$

Letting $n \to \infty$ and using above results, we get

$N(z - Bz, kt) \geq \min\{1, N(z - Bz, t)\}.$

$N(z - Bz, kt) \geq 1$

which implies that

$z = Bz.$

Hence

$Az = Bz = Sz = Tz = z.$

Thus, z is a common fixed point of A, B, S and T.

Similarly, we can prove the theorem when T is continuous.

Now, suppose A is continuous and the pair (A, S) is compatible, we have

$ASx_{2n} \to Az, A^2x_{2n} \to Az$ and $SAx_{2n} \to Az.$

Step 5. Putting $x = Ax_{2n}$ and $y = x_{2n-1}$ in (2.2.1), we have

$N(A^2x_{2n} - Bx_{2n-1}, kt) \geq \min\{N(SAx_{2n} - Tx_{2n-1}, t), N(A^2x_{2n} - SAx_{2n}, t), N(Bx_{2n-1} - Tx_{2n-1}, t), N(A^2x_{2n} - Tx_{2n-1}, t), N(SAx_{2n} - Bx_{2n-1}, 2t)\}.$

Letting $n \to \infty$ and using above results, we get

$N(Az - z, kt) \geq \min\{N(Az - \mathrm{z}, \mathrm{t}), 1\}.$

$N(Az - z, kt) \geq 1$

implies that

$Az = z.$

Step 6. Since $A(X) \subseteq T(X)$, there exists $v \in X$ such that $z = Az = Tv$. Putting $x = Ax_{2n}$ and $y = v$ in (2.2.1), we have $N(A^2x_{2n} - Bv, kt) \geq \min\{N(SAx_{2n} - Tv, t), N(A^2x_{2n} - SAx_{2n}, t), N(Bv - Tv, t, N(\mathrm{A}^2x_{2n} - Tv, t), N(SAx_{2n} - Bv, 2t)\}.$

Letting $n \to \infty$ and using above results, we get

$$N(z - Bv, kt) \geq \min\{N(z - Bv, t), 1\}.$$

$$N(z - Bv, kt) \geq 1$$

which implies that

$$z = Bv.$$

Since B ant T are compatible and $Bv = Tv$ implies that

$$N(BTv - TBv, t) = 1.$$

Therefore

$$Bz = BTv = TBv = Tz.$$

Step 7. Putting $x = x_{2n}$ and $y = z$ in (2.2.1), we have $N(Ax_{2n} - Bz, kt) \geq \min\{N(Sx_{2n} - Tz, t), N(Ax_{2n} - Sx_{2n}, t), N(Bz - Tz, t), N(Ax_{2n} - Tz, t), N(Sx_{2n} - Bz, 2t)\}$.

Letting $n \to \infty$ and using above results, we get

$$N(z - Bz, kt) \geq \min\{1, N(z - Bz, t)\}.$$

$$N(z - Bz, kt) \geq 1$$

which implies that

$$z = Bz.$$

Step 8. Since $B(X) \subseteq S(X)$, there exists $w \in X$ such that

$z = Bz = Sw$.

Putting $x = w$ and y = z in (2.2.1), we have

$N(Aw - Bz, kt) \geq \min\{N(Sw - Tz, t), N(Aw - Sw, t), N(Bz - Tz, t), N(Aw - Tz, t), N(Sw - Bz, 2t)\}$.

Using above results,we get

$$N(Aw - z, kt) \geq \min\{1, N(Aw - z, t)\}.$$

$$N(Aw - z, kt) \geq 1$$

which implies that

$$z = Aw.$$

Since A and S are compatible and $Aw = Sw$ implies that

$$N(ASw - SAw, t) = 1.$$

Therefore

$$Az = ASw = SAw = Sz.$$

Hence

$$Az = Bz = Sz = Tz = z.$$

Thus z is a common fixed point of A, B, S and T.

Similarly, we can prove the theorem when B is continuous.

Uniqueness.

Let w be another common fixed point of A, B, S and T, then

$$\omega = A\omega = B\omega = S\omega = T\omega.$$

Putting $x = z$ and $y = \omega$ in (2.2.1), we have

$N(Az - B\omega, kt) \geq \min\{N(Sz - T\omega, t), N(Az - Sz, t),$
$N(B\omega - T\omega, t), N(Az - T\omega, t), N(Sz - B\omega, 2t)\}$.

Using above results, we get

$$N(z - \omega, kt) \geq \min\{N(z - \omega, t), 1\}.$$

$$N(z - \omega, kt) \geq 1$$

which implies that

$$z = \omega.$$

Therefore, z is unique common fixed point of A, B, S and T.

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